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ABSTRACT

This book is about children's needs for intellectual rigor, as seen through the work and discourse of five undergraduate elementary education students who constructed a number system in base five and then reflected on their experience in the light of what they know about young children's mathematics. The deeper purpose of developing a number system is to explore precisely what it means to perform an act of learning. It is concluded that people reconstruct their culture's mathematics as they learn it. (Contains 49 references.) (ASK)

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FIVE WOMEN BUILD A NUMBER SYSTEM

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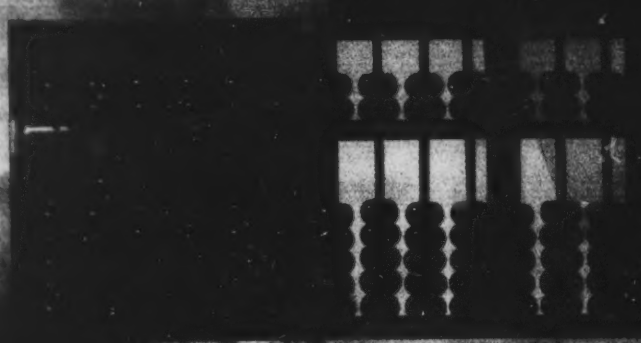

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FIVE WOMEN BUILD A NUMBER SYSTEM

MATHEMATICS, LEARNING, AND COGNITION

Carolyn Maher and Robert Speiser, Series Editors

Five Women Build a Number System
by Robert Speiser and Chuck Walter, 2000

FIVE WOMEN BUILD A NUMBER SYSTEM

by
Robert Speiser
Rutgers University and
Brigham Young University
and
Chuck Walter
Brigham Young University

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Foreword

GROWING A LOVE OF MATHEMATICS

Bob Speiser and Chuck Walter extend the work—begun by Bob Davis, Carolyn Maher, and Speiser—on deep symbolic interpretations of manipulation of concrete objects in numerical settings. Manipulatives have a mixed reception in mathematics learning and instruction: their inherent strength, based on tactile and visual perception, has often been negated by a learner's focus on nonessential perceptual properties. In simple terms, a learner using manipulatives seems to need to know what to focus on before they can focus on it.

Bob Speiser has done some beautiful work with Carolyn Maher on the highly productive and generative ways in which concrete material can provide entry points into significant mathematical thought. In this memoir, Speiser and Walter continue to investigate how concrete material can act as an organizer, grounder, and essential stimulator, for thinking about numbers in a deeper sense than is usually done. They circumvent many of the representational difficulties with concrete material by focusing on the *reasons* that the participating students believe what they do.

Speiser and Walter, in their discussion chapter, focus on the blocks used by the five women teachers, on information gained through working with them, and on how this information might be organized. Their work reminds me of a conversation with Bob Davis, at the recent Psychology of Mathematics Education Conference in Finland. There, a participant in a discussion had described information as residing in a bead frame, or abacus. For me, information does not reside in a physical apparatus but in the probability distributions that we place on our perceptions of the apparatus. We lose information by gaining order, by actively structuring the apparatus. This might seem like a semantic quibble, but I think it runs deeper. Throughout physics, for example, it is com-

mon to think of probabilities as given by the physical environment, rather than as *a priori* measurements carried out on the environment by our minds. However, it is through our unconscious ability to assign probability measures—how else could we make our way in an uncertain world?—and our ability to order the physical world that we gain an understanding of it. This is what we see with the five women teachers. They begin with relative chaos—a bunch of blocks to represent numbers. In the beginning most possibilities seem equally likely (or equally unlikely). Then by acting and reflecting they lose an enormous amount of information by way of imposing a stunning order on combinations of blocks. So, for me, the information doesn't reside in the blocks. It resides in the chaotic mess that the blocks present, and the learner's task is to order the physical situation to remove this chaos: to bring about order through the loss of information. Perhaps a small point, but I think a critical one, and essential to a constructivist view of learning: it is through our purposive and intelligent actions on our environment that we learn to act productively. It is through discussion of these intelligent, purposive actions that we learn to build general schemas, and to make our actions conscious and generalizable.

Mathematics is a cultural phenomenon. It does not reside in one person's mind, nor does any one person have a full understanding of the manifold ways of mathematical thought. This can create a dilemma in a beginner's mind if they agree that mathematics must, necessarily, be constructed by a learner. Naively, this constructive aspect of learning is often confused with discovery learning, in which students are led, or stimulated, to discover aspects of mathematics. Piaget was well aware that such a view is nonsense, in light of the long, deep cultural history of mathematical thought.

Those of us involved in mathematics teacher education, particularly elementary teachers, know how important and powerful a sense of ownership can be in turning prospective teachers around in their attitude to mathematics. Most trainee and newly qualified teachers wonder "How am I going to teach mathematics?" What they rarely wonder is "How do I learn mathematics?" This is not surprising, for they are concentrating on what will be, or has recently become, their job: to teach mathematics. But the ultimate aim of teaching is student learning, and teachers need good, positive, memorable experiences of their own learning in mathematics. Sadly, many mathematics teachers have not had such experiences. Speiser and Walter give a detailed and graphic account of how five women came to focus on their learning, and so gain a deeper understanding of what it means to engage one's mind fully with mathematical thought.

One of Speiser and Walter's students, Katie, encapsulated for me the value of this form of engagement with mathematics when she said: "I think that if all children were taught mathematics in this manner from the beginning of his or her education, the love of mathematics would grow in schools. Students wouldn't be so confused. They would understand why."

May Bob Speiser, Chuck Walter, and their colleagues and students help this understanding and love to come to pass.

Gary Davis
Centre for Research in Mathematics Education
University of Southampton, England

Preface

This book is about children's needs for intellectual rigor, as seen through the work and discourse of five undergraduate elementary education students who constructed a number system in base five and then reflected on their experience in the light of what they know about young children's mathematics. These preservice teachers encountered an important truth: that mathematics in the making—as opposed to mathematics ready-made—is tough, exciting, and complex enough to elicit deep, dramatic changes in its learners, whether children or adults. Their experience runs counter to stock rhetoric about how education students might explore and use the mathematics they will teach. Their perceptions about their mathematics—and about themselves—can challenge all of us to look more closely at whatever preconceptions we may hold about our students' personal and intellectual potential.

For example, we have heard that learning builds current knowledge incrementally, based on a store of past experience, like a pile of sand increases with each new shovelful. What might that mean, specifically, for the "content course" in mathematics for preservice teachers? The metaphor of "content" deserves examination here, as does the implied suggestion that something increasing needs to be contained.

For these five students, and for us, however, mathematical discovery, entails rebuilding *prior* understanding based on *current* experience, in a process that seems neither gradual nor cumulative. Instead, our students focus on fundamental doubts and disagreements in which mathematical argument means precisely what the term suggests: the struggle for assertions that compelling reasons can support, and the concurrent struggle for compelling reasons to support them. As human performances, discovery and argument define their actors through the dramas they enact. As we build ideas, we build ourselves, partly by tearing down our past constructions, based on current insights, often delivered with forceful effort by our peers.

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On the surface, this book describes how five elementary education students built a number system; but its deeper purpose is to explore precisely what it means to *perform* an act of learning.

Our intellectual forebears include Bob Davis, Victor Turner, Clifford Geertz, and William James. From Davis we have learned the value of an unrelenting emphasis on precisely how specific learners work with information. From Turner we have seen how social action and cognition are deeply intertwined, through clashes of discourse in which ideas are fought for, are modified, and sometimes help to change society. We share with Geertz a sharp sense of the challenge of describing social processes, of how people act, in part by building and revising what may best be viewed as narratives in which the personal and social deeply interpenetrate. Indeed, in this book we build interpretations, based on narratives about what has happened, to give some sense of how things happen in the contexts we describe.

We share with James the important recognition that concepts are immutable *by their very nature*, and can therefore trap us:

> Each conception...eternally remains what it is, and never can become another. The mind may change its states, and its meanings, at different times; may drop one conception and take up another: but the dropped conception can in no intelligible sense be said to *change into* its successor. The paper, a moment ago white, I may now see to have been scorched black. But my conception "white" does not change into my conception "black." On the contrary, it stays alongside of the objective blackness, as a different meaning in my mind, and by so doing lets me judge the blackness as the paper's change. Unless it stayed, I should simply say "blackness" and know no more. Thus, amid the flux of opinions and of physical things, the world of conceptions, or things intended to be thought about, stands stiff and immutable, like Plato's Realm of Ideas. (James, 1983, pp. 436–437)

One implication of James's view is that learning often builds completely *new* concepts; the changes that we wish to follow, therefore, cannot be wholly incremental or continuous. In particular, we need to recognize our learners' hesitations, doubts, crises, and confrontations as significant; hence, we need to study these with focused care and sympathetic understanding.

Our work, therefore, needs first to be *cognitive*, in the sense that we will need to focus very carefully on how both our own and our students' knowledge might be represented, and then *narrative*, in the sense that each learner tells such a story through his or her words and actions and that such stories mesh and overlap to give composite pictures of a social process in which learning, (in this case, mathematics learning) takes place because it is, and has to be, performed.

We are especially grateful to Bob Davis. His ideas helped set ours in motion, and his interest spurred us to maintain our efforts despite protracted, daunting opposition. Certainly his fine paper (Davis, 1987) on mathematics as performing art, which helped to open the way to our own work on classroom drama (Speiser

& Walter, 1997), a predecessor and companion to this monograph, deserves to be much better known. Our debt, however, is both greater and more fundamental. The classroom experiment that we discuss in detail here grew directly from our efforts to understand Davis's approach to representations, through the postulation of cognitive structures (Davis, 1984), in order to interpret learners' actions. Indeed, and perhaps most fundamentally, we share Davis's underlying motivation for investigating representations in the first place: taking learners seriously, as original and productive thinkers. In Davis's words, describing the ideas of the Madison Project:

> If we invite students to think, we have the obligation to take their ideas seriously. I have seen a videotaped lesson (not made by the Madison Project) where a teacher asks a class to invent a way to add fractions. The students have, on their tables, a collection of blocks of various shapes and colors. The teacher sets no criteria, nor any particular context, for this task.
>
> A student responds by proposing the algorithm of adding numerators to get the new numerator, and adding denominators to get the new denominator. The teacher rejects this as wrong.
>
> I would argue that the teacher is wrong. Since no specific context was specified, the task was tantamount to "Invent some sensible way to use fractions to model the real world in some way that relates to 'combining' or 'putting things together.'" The student has done this very successfully and very cleverly (as, indeed, he demonstrates, using the wooden blocks). The teacher ought to have recognized this and acknowledged it. (Davis, 1992, pp. 349–350)

Our conversations with Davis about cognition and learning began in earnest soon after we submitted our first paper about calculus (Speiser & Walter, 1994) to the *Journal of Mathematical Behavior* and continued through the next three years. As we complete the final draft of this book, just two weeks after his unexpected death, we hasten to record our gratitude for the supreme good fortune to have known him, to have learned from him, and to have worked with him.

The organization of this book reflects the central influence of representations as well as our particular approach to classroom narrative. At the center is an uncut transcript, stripped down to words and actions, which we view as a performer's script. Before this text, we give a cognitive discussion in which frames (Minsky, 1975, 1985; Davis, 1984) provide important ways to talk about the concrete images that our students use.

First and foremost, we have striven to do justice to our students' work and insight. As we learned from Davis, and our other forebears, we have also learned from these five women. The many pages that we offer here give just one look at what five energetic students did in less than 20 minutes. Other viewpoints are possible, based on what we, and they, may later come to think or know. Indeed, time's arrow flies. Large parts of what has come before are wiped away, or rebuilt, reshaped as new constructions meet and help to constitute current experience. We

are reminded of a suggestion from linguistics that social interaction triggers drastic, sudden, fundamental, and specific inner growth. How can this happen?

> The language each person acquires is a rich and complex construction hopelessly underdetermined by the fragmentary evidence available [to the child]....Nevertheless individuals in a speech community have developed essentially the same language. This fact can be explained only on the assumption that these individuals employ highly restrictive principles that guide the construction of grammar. (Chomsky, 1975, pp. 10–11)

Somehow, it seems, we must know *how to do* this building, based on an endowment that is partly innate, with awesome implications for each learner's human worth and also learned socially, with similarly awesome implications for the culture that we help construct.

We owe special thanks to Carolyn Maher and Alice Alston, who watched our tapes at Rutgers University just weeks after we made them, and immediately became our students' advocates as well as ours. At our own university, the work our students did brought forth support that changed our own perceptions of what we and others could accomplish through joint action. We are especially grateful to several administrators who reached across jurisdictional boundaries with grace and insight to support our efforts: Bill Hays, Larry Christensen, Bob Patterson, Russ Osguthorpe, Don Jarvis, Bonner Ritchie, and Paul Cox.

We are deeply grateful also to our research assistant, Tara Lewis, for her sharp eyes, knowing ears, and careful, energetic mind. She ran our camera, typed our transcripts, helped us in thousands of ways. This book and her thesis (Lewis, 1997) emerged from the same crucible, a year-long conversation among the three of us about what our students might become, based on our choices and their actions.

1

Introduction

> You make a great, a very great mistake, if you think that psychology, being the science of the mind's laws, is something from which you can deduce definite...methods of instruction for immediate schoolroom use. Psychology is a science, and teaching is an art; and sciences never generate arts directly out of themselves. An intermediary inventive mind must make the application, by using its originality...The teacher's attitude toward the child, being concrete and ethical, is positively opposed to the psychological observer's, which is abstract and analytic. (James, 1939, pp. 7–8)

In this book we explain how five young women—elementary education students in a mathematics course for future teachers—built a number system, including an arithmetic, in less than two hours, between 10:00 and 11:45 am, on Wednesday, October 11, 1995. By their own testimony, this accomplishment transformed the way they looked at mathematics and so transformed the way they saw themselves as thinkers. Here, for example, is Holly, one of the five women, writing in her journal eight days later:

> October 19, 1995
>
> Recently we have been working on creating our own mathematical system that does not utilize the numbers—one, two, three and so on—that we are familiar with. This task was difficult, especially at first. It is hard to step beyond what we are familiar with and comfortable with. However, when my group invented our own number sys-

> tem, it was exciting to come to a greater understanding of mathematics. In developing a new system, we had to examine exactly what adding, subtracting, multiplying, & dividing are. From my understanding so far:
>
> ADDING is counting the total amount in two or more groups.
> MULTIPLYING is adding certain amounts repetitively.
> SUBTRACTION is counting backwards when something is taken away.
> DIVIDING is subtracting a certain amount repeatively [sic].
>
> My understanding of subtraction and division need to become more complete. I comprehend the "gut reality" of addition and multiplication more than subtraction and division. I will continue working on these so my understanding is solid.

What Holly calls a "gut reality" we see as a powerful abstraction, a decisive step beyond received numerical procedures. The change in Holly's thinking, as she describes it here, is thus a qualitative change, a change in kind. As Holly emphasizes, this change—this discovery—was personal (by and for herself, expressed in the first person singular) as well as social (by and for her group, first person plural). It came about through careful work with concrete objects.

To understand what happened, we examine in detail how each group member built and represented her understanding, both for herself and through her interaction with the other four group members. For us, the challenge to construct an explanation seems, even in retrospect, quite daunting. Both the mathematics, as we saw it, and the social process interwoven with that mathematics seem to step outside established categories, demanding reappraisals that can still surprise us. In this initial section, we first sketch background and motivation and then offer an overview of the main strands of our analysis.

THE COURSE

In the autumn of 1995, we began to teach two parallel sections of Mathematics 305—the mathematics course for future elementary school teachers at Brigham Young University—not through conventional direct instruction but as a two-semester classroom teaching experiment in which we and our students worked as partners in a common inquiry. Our purpose was to learn how education students might construct, for themselves, the mathematics they would later teach.[1]

Each section, with about 17 students, met twice a week, for two hours each time. The students worked in assigned groups of five or six, maintained through the semester. We taught by offering our students tasks that aimed at fundamental issues, posed so as to require significant interpretation, exploration, and discussion, even simply to begin.

Like Deborah Meier (1996), we see teaching as mostly listening, and learning as mostly telling. Hence, we did as little telling as we could, sharpening our means for thoughtful listening instead. In particular, we videotaped each class,[2] took field notes, worked the students' tasks ourselves, discussed our own responses,

and took further notes. We focused our classroom practice on inviting students to explore specific, concrete tasks in detail, sometimes for several class periods, and then to present and justify what they had found. We planned each class based on field notes, headnotes (Sanjek, 1990), and weekly viewings of selected tapes of recent student work. Hence, our planning often took the form of writing task designs as direct, concrete responses to our students' efforts and ideas.

By the fifth week, our students had explored in two main areas. The first was to invent new algorithms. The second, strange as it might seem, was to count block towers.

For algorithms, we invited our students to devise ways to use Cuisenaire rods, squared paper,[3] and base 10 Dienes blocks to add, subtract, multiply, and divide whole numbers. We asked our students to explain their work in detail, especially why they thought that the results that they obtained, often through original, non-standard algorithms, were right. Exactly what, we often asked, do you *mean* by adding, or by multiplying? For a student who defines arithmetic through standard procedures, this question evidently poses an important challenge. Here, one week before the number system task, Holly reflects on her work, in which she had invented novel arithmetic algorithms, based on experiments with graph paper, rods, and base 10 blocks:

> October 4, 1995
>
> This past week, I have been impressed by the excitement that comes when devising your own algorithms. I think that if a child can create his or her own method to solving a problem, and still be arithmetically correct, they have a firm understanding.

While she still grounds arithmetic in numerical procedures, Holly has begun to think about how alternative procedures might be justified. In a breakthrough, just one week later, she will help to build the key abstraction that would generate, and justify a wide variety of algorithms for herself as well as others in the group.

Our students' breakthrough, as we see it, was the product of intense group effort over several weeks, beginning with a strong encounter with young children's thinking. The task that led to that encounter was to count block towers.

> How many different towers three blocks high can be built with Unifix cubes in two colors? How do you know that you have all the towers? How do you know that you have counted each tower exactly once?

This is precisely the task that Maher, Alston, and Davis (1992) offered to the fourth-graders in Kenilworth, New Jersey, who are now known as the "Gang of Four." Even towers three blocks high—there are just eight—allow drastically different conceptual organizations, which do not map easily to one another.[4] We chose this task not simply for its fundamental character,[5] but especially because of its clear demand for explanations that can lead to proofs (Maher & Speiser, 1997a, 1997b).

Introducing questions about towers, we aimed to focus class discussion on the need for reasoning and justification. After our students had worked carefully on building explanations for their findings, we showed them a taped discussion by the Gang of Four, age nine, of the same task. These children's efforts, and the students' discourse after seeing those efforts, would bring our classroom to a full, dramatic stop. The Gang of Four discovered not simply proofs but also (Maher & Speiser, 1997a, 1997b) the *idea* of proof. Our own students' discussion now opened to a wider range of questions. In this excerpt, Marci, who sat next to Holly and often shared ideas with her, reflects at the end of the semester:

> The first time we used unifix blocks, I didn't know what to think. When we were given a task, I immediately tried to think of what formula would help me figure out the answer. When we first worked with groups and had to find answers to problems, we all tried to think of what the problem had to do with our previous learning of math. Along the course of the semester, our group started to look at math in a different light. I knew that the problems we did had simple enough algorithms to figure out the answer, but it was more interesting to use the blocks or rods and really look into the heart of the problem.

Here Marci, like Holly, anchors an *abstraction* by exploring concrete objects, which she uses to recall and then to reconsider past experience (Maher & Speiser, 1997a, 1997b). This abstraction, supported by its concrete anchor, was precisely what she finds "more interesting," and hence her personal response reflects a deep change in how she thinks about her mathematics. What does it mean, concretely and precisely, to reach the heart of an important problem?

We have come to see the Gang of Four tape as a fiercely argued discourse, first about the particular patterns and organizations that gave raw materials for the proofs that the children built, and then about the role of patterns, more abstractly, in the construction of mathematical arguments. For our undergraduates, themselves debating what may well have been their first attempts at detailed mathematical justification, the protracted arguments through which the Gang of Four discovered the idea of proof have become metaphors, to which our students will refer repeatedly, for what most children, given good environments, can achieve, and, by extension, for our students' own potential.

Marci and Holly, each in her own way, describes key steps toward a significant abstraction. These steps, as with the Gang of Four, were triggered by the need for proofs. How can two different algorithms, for example, both be proven right? As Holly and Marci found, this question asks for underlying principles. On October 4, after she had built a novel algorithm for multiplication using base 10 blocks, Holly justified her new algorithm by representing it symbolically, in such a way that each step could be mapped to a corresponding step of a standard procedure. How did this correspondence come about? What might its underlying insight lead to?

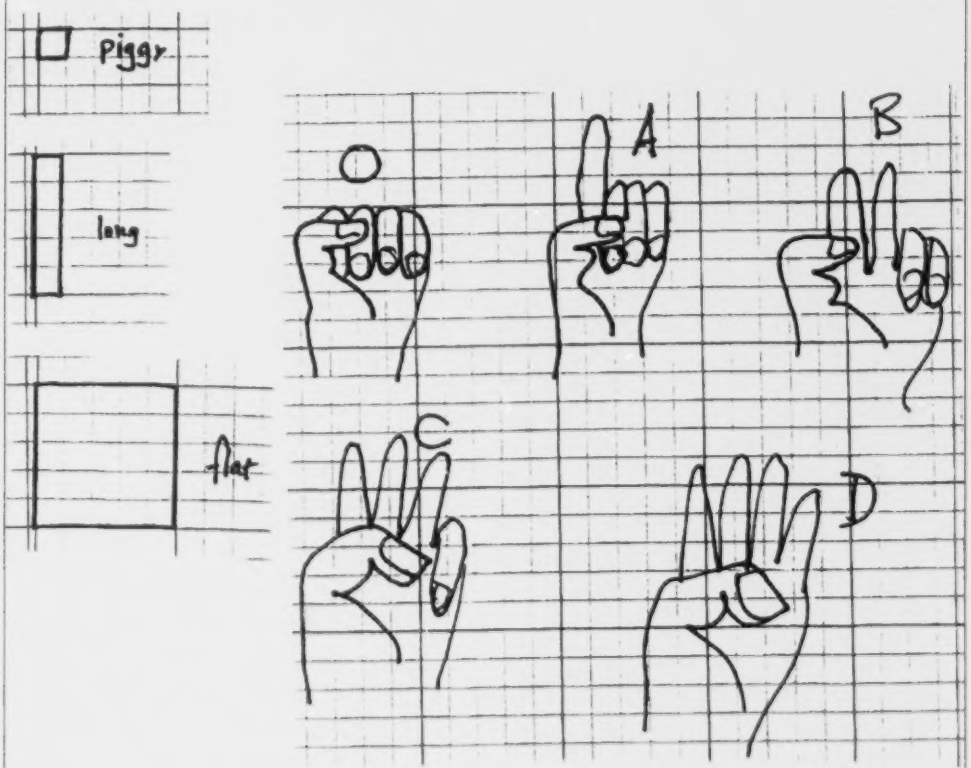
To us, the construction of an explicit isomorphism between two procedures looked exciting, not simply as a proof but also as a probe into procedural arith-

To: Mathematics 305
From: Professors Speser and Walter
Subject: Appeal to Your Imagination
Date: October 11, 1995

Our goal is to create a brave, new world, in which we can count, write numbers and do meaningful arithmetic, without ever leaving that world. Are you ready?

Task: In each group use the blocks below, including "solids" if you wish, to come up with a number system that only uses the symbols 0, A, B, C and D. Use the blocks to help develop and explain your system. Your number system should represent any number, no matter how large, and allow you to add, subtract, multiply and divide.

Rules: Never even think in base 10. Always express yourself with blocks, with the symbols 0, A, B, C and D, or with the gestures associated with the symbols.



Adapted from *Reconstructing Mathematics Education* by D. Schifter and C. Fosnot

FIGURE 1.1. The task.

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metic. In our planning discussions, looking at our students' work with base 10 blocks, we suspected that the all-too-tight connections to place values and procedures in base 10, which so many of our physical materials (not just our blocks) had made available, now needed be wedged away. The time had come, in other words, to challenge the stock story (Speiser & Walter, 1997) that multiplication is, at bottom, a procedure.

THE RESEARCH INTERVENTION

Our intervention for the decisive session on October 11 was to pose a task. In Figure 1.1, we reproduce this task, exactly as our students saw it.

We varied in several directions from the well-known version, designed by Schifter and Fosnot (1993, pp. 41–62) for workshops with inservice teachers, but we shared key motivations with these authors. We designed our task to be less constraining, in the sense that the physical and symbolic representations offered by our design were less tightly coupled. We hoped that one effect of this relaxed constraint was that our students would show us more about their own criteria for making choices, due to the wider range of possibilities.

In our view, success in this task should stem, at least in part, from our students' own construction of criteria for telling when they had made important headway. Our task removed the whole base 10 environment so as to force a search for new criteria on which to base truth claims. On what experience might these new criteria be based? Loosening the task, we gave our students very fundamental choices, beginning with decisions about what the given symbols ought to mean.

When we passed the task assignments to our students, their immediate response was to engage in rapid, anxious, highly animated conversation. About a quarter-hour into this exploratory discourse, Tara Lewis, our research assistant, sensed something special happening in one group, without a clear idea of what:

Mandy: We're supposed to do it with our blocks, the blocks...
Holly: Because you could write...
Laurel: We could write them.
Holly: Right.
Laurel: We don't have to just use the blocks. We could use a piece of paper and write ADD.
Holly: You could use your fingers, you could use your writing, or you could use your blocks.
Katie: Okay. Are these supposed to represent the blocks? I mean, where it says, "In each group, use the blocks below, including 'solids' if you wish, to come up with a number system that only uses the symbols O, A, B, C, and D." Okay. So are we supposed to represent these *[pointing to the drawn blocks]* with a symbol *[pointing to the symbols on paper]*?
Marci: Maybe just something like that. What if we used the blocks as the numbers

> and these as the symbols? As like these are the numbers that we use [pointing to the blocks]. A would be "add," B would be "subtract," C would be "divide," and "D" would be "multiply" *[making the appropriate hand signals as she explains]*. Getting the blocks, saying here's one block then *[making the signal for A]* that would be adding? Maybe?
>
> Laurel: I don't understand.

Tara moved her camera close and then began to film this group nonstop.[6]

When we watched the full tape some days later, we sensed a major disagreement and then, indeed, a *drama*, well before we came to grasp our students' subtle, complex interactions. As we watched the tape repeatedly, to build and verify its transcript, the dramatic process came to seem quite fundamental, even for the roughest understanding of our students' thinking. In this book, we document in detail how our students built their representations, reasoning, and verbal presentations in order to conduct, quite literally, an *argument*, in which far more than simply mathematics seems to be at stake.[7] By the same token, our interpretations will conduct a further argument. The two arguments run parallel; hence, our own cognitive process—where we focus our attention, how we represent what we have seen—deserves examination and discussion, too.

THEORETICAL FRAMEWORK

We see information and experience as triggers for—and, at the same time, products of—theories, which are held, discussed, doubted, affirmed, and argued over, in a complex personal and social process. In particular, here we view arithmetic (both numbers and methods, jointly) as emerging from two levels of experience: the structuring of numbers and the structuring of concrete objects. This structuring takes place within a larger context of perceptual and structural investigation. In this instance, numbers will be "built" to reveal as well as to embody structures that will make sense personally, in the sense that they have meaning to their builders.

On the one hand, a theory is an engine of perception. On the other hand, a theory is a tool for winning battles. What structure? Who decides? And on what grounds? Our five students grasp these issues clearly as they argue, and they argue by constructing and then reasoning from symbolic representations (Davis, 1984), which they build both for themselves and for each other. In practice, we infer perceptions, and constraints upon perception from the way our students used their representations, in detail, to support specific arguments.

Here, given our course design, we consider theory building by learners in two layers: a foreground and a background. In the foreground, we build a narrative, which follows our five students closely. In the background, we consider the impact on our students of the Gang of Four,[8] who became, by videotape, guests in our classroom, just before the foreground action we selected for analysis.[9]

To build the foreground narrative, we interpret our students' arguments as attempts to deconstruct and then rebuild parts of their past experience. As in related work (Maher & Speiser, 1997a, 1997b), we use the metaphor of text. A learner, in effect, communicates to other learners, and to us, by composing or discussing texts. Past experience is often retrieved as text, to be reconsidered, commented on, and, often, rewritten. For us, a text is sometimes written down on paper or produced as spoken discourse, but not always. Further, we distinguish text, in part a product of interpretation, from whatever internal representations might produce it (see Chapter 14 in Davis, 1984).

In this vein, we can view the Gang of Four tape itself as a text, more precisely as a background text, on which we or our students may comment. What the Gang of Four tape *represents*, however, will be seen in further texts, built, for example, by our five students, by the research team at Rutgers who filmed the original, by the Gang of Four themselves (Maher & Speiser, 1998), or by us. For example, we might read this tape as the story of one child, Stephanie, who, battling for her personal idea, became a metaphor both for the Gang of Four's discovery of proof, and for the demand, which triggered that discovery, for clear, convincing explanations.

We seek to ground descriptions of cognition in narrative because, fundamentally (Speiser & Walter, 1997), we see cognition as unfolding through performance. In particular, the narrative we build tracks dramatic discourse (fighting for the floor, proposing challenging test cases, confronting a dominant discourse), in order to help locate fundamental mathematical issues. They are fundamental in the sense that they evoke important conflicts—conflicts that hinge on sense and meaning in compelling ways and which therefore *demand* clear resolution.

To provide enough detail to explicate even the key events, we have constructed *frames* (Davis, 1984), first to make sense, line by line, of our students' texts, and then, in similar detail, to build texts of our own. Frames are data structures, that capture information in precisely specified ways and hence make that information available for interpretation and analysis.

> We can think of a frame as a network of nodes and relations. The "top levels" of a frame are fixed, and represent things that are always true about the supposed situation. The lower levels have many *terminals*—"slots" that must be filled by specific instances or data. Each terminal can specify conditions its assignments must meet. (The assignments themselves are usually "subframes.") Simple conditions are specified by markers that might require a terminal assignment to be a person, an object of sufficient value, or a pointer to a sub-frame of a certain type. More complex conditions can specify relations among the things assigned to several terminals. (Minsky, 1975, p. 212)

For us, frames help to represent mathematical information as it might be used and understood. In this view, learners build frames to represent the knowledge they are building, based on personal experience, with enough precision to sustain

detailed—and sometimes hostile—critical examination. In large measure, these frames represent *our* understanding of our students' personal constructions.

As Noddings (1990) observes, to say simply that learners build their knowledge hardly tells us anything at all until we say, with some precision, what is being built, by whom, and why. To say why our students work as they have done, and not in other ways, has got to be, as the expression goes, another story.

In our view, learners build their knowledge in order to reach goals. For us, these goals emerge from texts, often implicitly, as we examine our learners' hesitations, doubts, experiments, and disagreements. As actors, our five students speak, write, draw, and move. To tell their story is to build interpretations of their words and actions. As distinct characters, each actor builds her own part over time—a strand of text within the larger drama. The *meaning* of that text, both for itself and as part of the larger action, therefore emerges only through detailed interpretation of each player's words and actions over time. Each player acts both for herself and as part of a larger action that is influenced by her participation.

The narrative emerges, therefore, as a composite of individual players' texts—public constructions—which first are built and then rebuilt. The reading of these texts and how they change depends fundamentally on detailed examination of the *mathematics* that the players build and hesitate and disagree about. To follow changes (text revisions) closely, specific details of our frames will matter greatly. Hence, for example, key parts of our frames—their "terminals" (or "slots")—need to be mutable, while other vital parts must remain stable. This distinction between mutable and stable features, drawn directly from our students' arguments and text revisions, sharply constrains the frames that we can build as well as how they might be used. Their mutability allows us to construct a narrative without abstracting from specific details that will matter to us.

Explicitly, our students' texts describe the building of a number system. Implicitly, however, through examination of their conflicts, hesitations, and experiments, they discover *mathematics* by discovering compelling arguments—the latter based, as with the Gang of Four, on making and investigating patterns.

Let us look more closely at the latter parallel. It was easy for the Gang of Four to reach a "consensus" that there must be exactly eight such towers, but not one of them would urge so weak a rationale. The need for *explanation*, as opposed to the mere recognition of a fact, would therefore drive that group's discovery of proof. Any such explanation, as all four children found, fundamentally depends on organizing the towers—which could be done in several ways, based on quite distinct ideas, but subject to strong constraints—and then on building proofs, each based on a particular organization. The significant discovery, however, triggered by one child's question, was about far more than how to mobilize a particular organization so as to build a proof.

> Jeff: I have a question. Why is everybody using patterns? Why do you have to go with patterns?

The children's discussion in response to this question proposed a strong epistemology: Without a pattern—that is to say, without a way of organizing—there can *be* no way to build an argument.

Similarly, for our five undergraduates it is not enough to build a number system and an algorithm, say, for finding products. There needs to be an *explanation*, which must link the design of the number system with the meaning given for the product, so that the proposed algorithm can be shown to build upon that meaning. It is especially interesting that the proposal to look for meanings through investigating patterned structures came directly from these learners themselves. For them the necessary structure grew, after a fierce argument, from organizing base five blocks. In other words, the blocks have functioned for these students as concrete mathematical objects, about which a *theory* first is built, and then applied to numbers. As the emerging theory is constrained, so will be the numbers.

For our students, as for the Gang of Four, personal acts of reading structures into objects and, hence, to some degree, *converting objects into texts*, will constitute the story. The details, which we need to treat with care, emerge from structures that we, the authors of this book, build, namely the frames we postulate to represent the objects, both physical and mathematical, which we and our students work with, but the *story* will be told through our interpretations of our students' actions, their performance, as a drama, both of people and ideas.

METHODOLOGY

Our students' work and our discussion of it represent one episode in a constructivist teaching experiment, in a sense quite similar to that of Steffe (1991). For Steffe, who focuses on clinical interviews with individual children, such an experiment is characterized by the way the researcher functions in the role of teacher, through a thoughtful, mutually responsive relationship with the learner, in which the researcher proceeds by building a map or model of "each child's mathematical knowledge," frames hypotheses in terms of that map or model, and tests these hypotheses "by initiating probes that might stretch the child to the limits of his or her understanding." As the child proceeds, the researcher's model evolves; hence, in cycles, it generates more probes, more interaction, and hence further construction, both by the child and by the researcher. Steffe emphasizes how the teacher, in this process, learns from the child, in the sense that the teacher builds models of the child's developing understanding. He writes with evident delight about those special moments when the child, significantly, chooses to lead the student–teacher interaction.

As teaching researchers, we also act in the role of active learners, in partnership with our students, but with the important difference that we worked, not with individual children but with a group of very active young adults, whose interactions in the segment we report (save for a brief comic interlude, in which one researcher beats a quick retreat) were entirely with one another. Further, our students' math-

ematical constructions were at times so complex that our own constructions entailed quite difficult decisions, often requiring months, to make good sense of what our tapes revealed, long after the events that they recorded were over.

As a consequence of this delay in our interpretive response, we *had* to teach, in Meier's (1995) words, mainly by listening, and then by building *provisional* models, primarily to pose new tasks rather than to intervene directly. We made these preliminary attempts at models on the fly, from what we saw on early viewings of our tapes, from our rough field notes, and from further notebooks that recorded mathematical experiments aimed at understanding what our classroom data seemed to show. The tasks we posed most certainly were "probes" in Steffe's sense, within a setting, as we have described, that did reflect key aspects of our models, but with the important difference that our own learning seemed far more tied, especially at first, to understanding what the *mathematics* was, in such a way that we could then, as a result, frame even preliminary guesses as to how to read our student's actions. In practice, then, we let our students work with little intervention, often for hours at a time, before inviting them, when they felt ready, to make formal presentations to the class.

In stepping back into the role of quiet, long-term listeners, we deliberately allowed our students leeway to design informal teaching experiments of their own, which they conducted on one another, basing their decisions on how *they* chose to represent one another's understandings. Because our listening often produced quite tentative, and even doubtful, models in the short run, our further tasks and interventions reflected the limitations of those models, often in very striking ways.

Like Steffe, although with different emphases, we aimed for "thick descriptions," in the way that Geertz (1973) uses the term, to go beyond mere records of words and gestures (whatever that may mean) to build narratives, to map qualitative changes over time. Our own probe led us to concentrate on building detailed structures to represent the *information* that our students used, in order to describe their use of it. These information structures, which we chose to build as frames, are not quite cognitive and not quite mathematical but, rather, as Minsky (1985) and Davis (1984) emphasize, they are something new: a way to represent our knowledge about mathematics *in the making*.

Our emphasis on text (Maher & Speiser, 1997a, 1997b) points to another central emphasis, in that we need to place and then maintain detailed attention both on how our students personally represent their information and then on how they choose to use it—revising and rebuilding—as they argue, hesitate, construct experiments that test both rhetoric and key hypotheses, and, eventually, find ways to build important new ideas, through conflict, error, and significant experimentration.

That we now see our students' work as drama, for example, both demonstrates and fundamentally requires the *evolving* nature of our work with models or our work with texts. Indeed, we posed our research tasks deliberately so as to leave open *to our students* how to shape the tone, the sequence, and the substance of their actions.

TOWARD A VIEW OF MATHEMATICS IN THE MAKING

For us, our students' mathematics (and more generally, school mathematics) is a subject in the making, which we seek to understand through joint research conducted with our students. Hence, we need to emphasize that neither the mathematics nor its social context may be assumed in advance. Instead, we need to understand them through close study of our students' actions, based on constructions that we build in order to make sense of them.

In other words, we favor what Bruno Latour (1987) calls *science in the making*, as opposed to *ready-made science*. To symbolize the difference, Latour imagines a timeline above which, at the present moment, a two-headed Janus face presides. On the left, facing the past, an older face represents ready-made science. On the right, facing the future, a younger face represents science in the making. They offer contradictory advice:

> Ready-made science: When things are true, they hold.
> Science in the making: When things hold, they start becoming true. (Latour, 1987, p. 12)

To study science in the making, as Latour describes it, means to study how black boxes come to be closed—that is, how hesitations, doubts, and disagreements come to be resolved, for that is how things come to hold. In this book, we study the course of one such disagreement in order to examine fundamental issues that the disagreement raises. In particular, we look closely at what will come to hold, and what will not, under our students' questioning.

> The [term] *black box* is used by cyberneticians whenever a piece of machinery or a set of commands is too complex. In its place they draw a little box about which they need to know nothing but its input and output. As far as John Whittaker is concerned, the double helix and [his computer, on which the double helix is displayed] are two black boxes. That is, no matter how controversial their history, how complex their inner workings, how large the commercial or academic networks that hold them in place, only their input and output count. (Latour, 1987, p. 3)

At the start of our experiment, base 10 numbers provide such a box, and we designed our task, as we explained above, to force it open.[10] It will open, indeed, with a shock. The ensuing action locates further boxes inside the one that we have opened, and these further boxes will become new objects for debate. In this second round, our students, by and for themselves, will design the probes that do so.

> Uncertainty, people at work, decisions, competition, controversies are what one gets when making a flashback from certain, cold, unproblematic black boxes to their recent past. If you take two pictures, one of the black boxes and the other of the open controversies, they are utterly different. They are as different as the two sides, one

lively, the other severe, of the two-faced Janus. (Latour, 1987, p. 4)

In particular, the frames we have designed reflect three purposes: first, to represent our own initial understanding, from which we planned our course and designed our interventions; second, especially through our frames' defaults and terminals, to highlight in fine detail the points that our students' own probes would disclose as pertinent about the issues they debated; and third, to ground our own discussions as we built and then rebuilt our own interpretations.

To study mathematics in the making, we must write, as best we can, from *inside* the process of construction. From the inside, the context and the content, both under construction, become important subjects of debate. They intertwine. To illustrate this central point, Latour (1987) cites James Watson's frank description (1968) of how he and Francis Crick decided to go public with their model for the structure of DNA, and then comments:

> 'Suspense,' 'game,' 'tone,' 'delay of publication,' 'awe,' 'six weeks delay' are not common words for describing a molecul[ar] structure. This is the case at least once the structure is known and learned by every student. However as long as the structure is submitted to a competitor's probing, these queer words are part and parcel of the very chemical structure under investigation. Here content and context fuse together. (Latour, 1987, p. 6)

Our students' mathematics and its context need to be inferred from later discourse. Indeed, from our perspective, that mathematics, together with the social context where that mathematics will be built, are (Latour, 1987, p. 13) *results*, not causes, of the way our students' doubts and disagreements will be settled.

> We will enter facts and machines when they are in the making; we will carry with us no preconceptions of what constitutes knowledge; we will watch the closure of the black boxes and be careful to distinguish between two contradictory explanations of the closure, one uttered when it is finished, the other while it is being attempted. This will constitute our *first rule of method* and will make our voyage possible. (Latour, 1987, p. 13)

On the one hand, we need to be on guard against assumptions that constrain either our students' mathematics or the social setting which that mathematics, in the making, helps to constitute. On the other hand, our narrative needs to address both the mathematics and its context as we and our students build and then rebuild them. Two further Janus heads (Latour, 1987) suggest that mathematics, as we build it, and its social context, as we seek to understand it, intertwine in very complex ways:

> Ready-made science: Nature is the cause that allowed controversies to be settled.
> Science in the making: Nature will be the consequence of the settlement. (Latour, 1987, p. 99)

> Ready-made science: Society is the cause that allowed controversies to get settled. Science in the making: A stable state of society will be the consequence of settling controversies. (Latour, 1987, p. 143)

On the basis of the tape we analyze below, we wonder what Latour might mean by "stable state." Near the end of our narrative, indeed, one student, Laurel, who helped fight—successfully and hard—to build a strong, emergent discourse, will draw back, nonetheless, from what might otherwise have been construed as a consensus. We have come to treasure her reluctance to endorse what may have been, for her, a superficial closure. We prefer to leave the scene exactly as our students played it, with remaining issues—which must, necessarily, remain unknown to us—still to be resolved. Does our story's ending, with its partial resolution *and* its hesitation, represent a "stable state"? For Laurel's sake, and for our own, we elect to leave this question open.

WHICH POSSIBILITIES? THE QUESTION OF CONSTRAINTS

The structures that our students build seem far from arbitrary. They satisfy design constraints that—as the group would learn through its debates—would sharply narrow what is possible. Similar constraints limit the way we build the frames we use to represent the objects that we and our students study. Consider Minsky's (1975) model for representing visual information about a room. It is harder than one might think to build successful frames, even for a task that looks simple. The frames, as Minsky shows, have to interact, through shared terminals, often in complicated ways. Constraints emerge, as we see here, directly from these interactions and their consequences:

> For visual scene analysis, the different frames of a system describe the scene from different viewpoints, and the transformations between one frame and another represent the effects of moving from place to place. For nonvisual kinds of frames, the differences between frames of a system can represent actions, cause–effect relations, or changes in metaphorical viewpoint. Different frames of a system share the same terminals; this is the critical point that makes it possible to coordinate information gathered from different viewpoints. (Minsky, 1975, p. 221)

That there are exactly eight towers in the Gang of Four task reflects constraints on how the towers must be built. The children's pattern building—this seems to be the point of Jeff's sharp question—probes those constraints in systematic ways. So do our students as they work with number systems. Even with the looser task that we gave them, the design requirements still—but only after further argument and exploration—force particular agreements, and the recognition of the complex impact of these conditions seems to be the axis on which our students' mathematical discussion turns.

For us, the mathematics—not just the cognitive development—emerges from the ways in which we and our students work with given constraints.

Minsky's program for artificial intelligence is well worth reviewing here. Its first step is to try to simulate, on machines, key processes of human thought. Its second step is to argue that the constraints upon such simulations are severe enough to imply that our bodies and our brains, which may be similarly constrained, should resemble their simulated counterparts in ways that might contribute to our understanding.

The frames we build to help us understand how to construct arithmetic by representing concrete objects, and then by using those concrete objects to constrain our number systems and arithmetic, are similarly constrained, and hence, in Minsky's sense, should therefore be helpful, even though we postulate them partly for our own convenience, just as Minsky does in his description of a room.

GUIDING QUESTIONS

To begin, we summarize three key assumptions. First, learners' mathematics builds fundamentally, in ways often not fully known to us, from reading *structures* into *things*. The choice of an organization for three-high towers provides one example. So does the construction of a number system. Second, our interpretations build from reading *frames* into our learners' *discourse*. In the spirit of Davis (1984), children's work with towers, based on how they organize them (Maher & Martino, 1996a, 1996b; Maher & Speiser, 1997a, 1997b) can be described quite fruitfully this way; so, as we shall see, can our students' arguments about the number system they construct. Third, both these readings seem highly *constrained* the moment we consider concrete *data*. Our research questions, therefore, center on investigating structures, readings, and constraints.

What do our students do, specifically, in order to pursue their goals? How do our tasks and the environment surrounding them support that process? How do our postulated frames support our understanding? How are the fundamental judgments made that will constrain and motivate the theories being built?

We may wonder, for example, why our students' texts make sense to us as well as to their authors. In the transcript we give below, the counterstory that Katie and Laurel tell rings true, first for their listeners and then for us, albeit in very different ways. For their listeners, Katie and Laurel forced others in their group to look quite deeply at the meaning of their mathematics. For us, in addition, Katie and Laurel worked hard to reveal, perhaps at some personal risk, underlying tensions in their group that prevented a solution not just of a mathematical problem, but also of further questions. What does it mean, in human terms, to seize the floor, produce decisive evidence, and then make space for new ideas? It is not enough just to be right, these women teach us. Indeed, we *need* to understand the impact on the learners of the battles and the reconciliations that enabled them to construct assertions with decisive power to persuade.

The larger implications of our students' work stem from the *metaphor* their story offers us, the story of five quite specific people who have found what Eleanor Duckworth

(1996) has called a *wonderful idea*. Strange to say, within one hour, our students' mathematical discovery became a neat black box for the people who constructed it, while the *story* of the way they built that box and closed it has occupied our own thoughts for nearly two years. Our final guiding question has to be about the closing of our own box, which the writing of this paper partly represents. What, precisely, do the constructions that we and our students jointly built suggest, about how curricula and classrooms might best be imagined, built, and thought about? We, like Laurel, decline a superficial closure here.

THE DATA: TAPE, TRANSCRIPT, AND ANNOTATION

We began with about 45 minutes of uncut classroom documentary footage. The first step was to produce a rigorously verified transcript of the full tape. We then selected 16 continuous minutes for analysis.[11] Later, for analysis, we broke the transcript into scenes and augmented the raw text to provide, in principle, a full performers' script.

The initial transcript was prepared by our graduate assistant, Tara Lewis. It was verified in three passes by all three of us, as interpretive hypotheses demanded more detailed examination of the transcript and its annotation. In addition to corrections of notated speech, these revisions often focused on nonverbal cues, including gestures, facial expressions, tones of voice, and body movements such as standing up and moving forward.

OVERVIEW

To help the reader understand how we thought about our course as well as the specific task design, we present a set of frames, then discussions of these frames and their potential uses. Our work with frames grows from a particular understanding of children's arithmetic, based strongly on research by many workers, much of it described in Ginsburg (1989). In particular, we use well-known physical materials in a variety of ways—some standard, others not. Much of our work depends on children's number folklore and on our students' own interpretations of the key materials.

Next, we provide the uncut transcript, organized in scenes, with players' directions and with diagrams of all the physical constructions that our students made, accompanied by very light analysis.

Following the transcript (which should be first read in its entirety), we give a scene-by-scene analysis, based closely on the frames that we have built, to provide local descriptions of the action—a drama that hinges on important mathematics.

Finally, we discuss some larger issues raised by our students' work; by our views of that work; and by related questions about mathematics, about personal cognition, and about how groups of people come to change their minds, resolving conflicts through significant discoveries. In this final section, we again consider mathematics in the making in order to focus on how schools and learners help to rewrite society, for good or ill, in the midst of conflict and uncertainty, day by working day. The key, as Duckworth (1996) says, is having wonderful ideas. The hard part, as our students teach us, is to stand, move forward, and insist on looking deep enough to find them.

2

Frames for Counting, Numbers, and Arithmetic

Our students argue about numbers, counting, and arithmetic. In this section, we attempt to sketch some of our own views of this mathematics, leaving for later a more detailed look into the students' explorations. Their work surprised us. That we chose to see their work as drama also came as a surprise.

To describe the mathematics that will be at issue, we take a point of view strongly espoused by Davis (1984), based on earlier research by Marvin Minsky (1975), Roger Schank (1973), and several others.[12] The central theme is to examine, in concrete detail, how we ourselves might represent mathematical knowledge cognitively. The position we take here, with Davis, is that we do so symbolically, often through the medium of particular representation structures called "frames."[13] For Minsky (1975, p. 212), a frame is a data structure, in particular a retrievable computational object, that represents a "stereotyped situation, such as being in a certain kind of living room, or going to a child's birthday party." Each frame also carries information—for example, about how to use the frame or about what might happen next. Most important, some of this information "is about what to do if these expectations are not confirmed."

Minsky sees a frame as a network of "nodes and relations." The "top levels," which are relatively stable, "represent things that are always true" about

the given situation. Below these levels, the frame becomes more mutable. At the lowest levels, there are what Minsky calls "terminals" or "slots" that must be filled by specific chunks of information; these vary from instance to instance. To provide these chunks of information, which we call "inputs," *instantiates* the frame.

When we retrieve a frame, it often is not clear, at least at the outset, how we might evaluate its inputs. In that case, at least three strategies seem possible. The first, which Minsky stresses, is to provide "default" values, perhaps seen as typical from past experience; these, as past experience should also teach, would need to be quite loosely held. The second, which Minsky also suggests, is to leave some inputs temporarily blank and then pose questions aimed at learning how to fill them. This second strategy proposes an elegant explanation for why we ask some of the questions that we do—namely, to locate inputs for frames. The third strategy is to assign values obtained from other, related frames, with a view toward learning later what will happen as a consequence. This approach could well, produce cascades of interlinked assignments, sometimes with haunting, unforeseen results. When Davis writes, "Obviously a default evaluation is, in effect, a reasonably shrewd guess" (1984, p. 46), he may regard our third strategy as a variation on the first. It will help us later to distinguish them.

Both Davis (1984) and Minsky (1975, 1985) provide rich, specific evidence to show how helpful frames can be. Our goal here is to provide, through several specific frames, good starting points for a description of how we, our students in the context of this drama, and perhaps schoolchildren and their teachers work with numbers, counting, and arithmetic.

While frames, at least in principle, are precisely given formal objects, we shall freely intersperse our "hard" descriptions of them with informal comments, examples, and asides, in order to make our underlying motivations at least reasonably clear. Perhaps these less formal components, written here as text (Maher & Speiser, 1997a, 1997b), provide what Minsky calls "information about how to use the frames." We are especially concerned with what we need to notice in order to count anything. Often, what we need to notice seems to be a possibility for structuring. This motivates our first new frame.

COUNTING CONSTRUCTION FRAME

This frame is a collage of subframes. These most likely vary from person to person, and even within a given person, over time. Here we present some examples of such subframes that seemed especially helpful as we thought about our first semester's work.[14, 15]

FIGURE 2.1. The binary partition.

A. Binary Partitions

Inputs: a set, denoted S, together with a partition of S into two disjoint subsets, denoted Q and R. We sometimes call Q the *first part* of S and R the *second part* (Figure 2.1).

The binary partition frame includes as a special case a set S together with a partition where R has exactly one element. This special case turns up so often that we might call it the "one-more-element subframe."

The binary partition subframe connects, for example, to whole number addition and subtraction. The one-more-element subframe should help us represent iterated processes, such as counting or, more powerfully, recursion.

B. One-to-One Correspondences

Inputs: two sets, denoted S and T, together with a given 1–1 correspondence between them. We might imagine the elements of S and T, respectively, to form the rows of an array, like that shown in Figure 2.2.

Here, one row represents the members of S, the other row represents the members of T, and the pairs of corresponding elements form columns. This picture is potentially misleading, however, in that we do not impose a particular order on the elements of S or T. All we have in mind is the correspondence between the elements of S and T, as given by the columns of this array.

The one-to-one correspondence subframe is useful for comparing sets, with or without counting.

C. Rectangular Arrays

Inputs: a set, denoted S, and a 1–1 correspondence between the elements of S and the elements of a given rectangular array, such as shown in Figure 2.3. (But if S is empty, so is the array.)

FIGURE 2.2. The elements of S and T arranged as rows.

FIGURE 2.3. A rectangular array that corresponds to S.

The given array can have any number of rows or columns. (If the array is empty, there will be none of each.) Informally, we can imagine the elements of S themselves as forming a rectangular array. The rows of the array partition S into disjoint subsets in one way, while the columns partition S in another way. These two partitions are included in the rectangular array subframe.

The rectangular array subframe connects, for example, to whole number multiplication.

D. Staircases

Inputs: a set S, together with a 1–1 correspondence between the elements of S and the elements of an array, of the form shown in Figure 2.4.

Here, each row, except the lowest, corresponds (vertically) to a subset of the row below it and partitions that lower row according to the one-more-element subframe. We say that two staircases have the *same size* if we can set up a 1–1 correspondence between them.

FIGURE 2.4. A staircase.

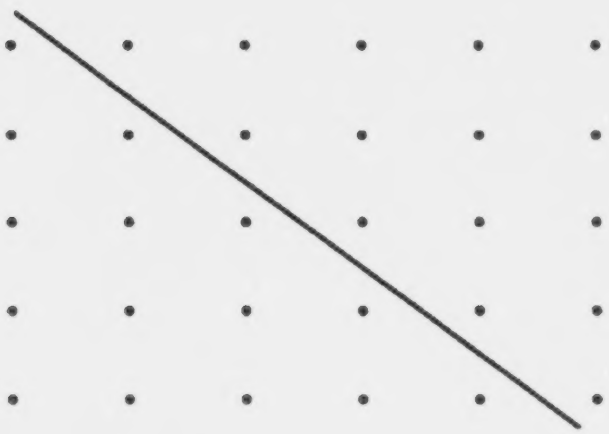

FIGURE 2.5. A double staircase.

The staircase subframe connects to sums of consecutive numbers, such as $1 + 2 + 3 + 4 + 5$, and is based on a well-known Cuisenaire rod construction of the same name.

E. The Double Staircase

Inputs: two staircases of the same size. The two staircases fit together to form a rectangular array (see Figure 2.5). The rectangular array, as in Figure 2.5, will always have one more column than it has rows.

The double staircase corresponds to a classic Cuisenaire rod construction of the same name. It allows us, without adding, to conclude, for example, that we have

$$1 + 2 + 3 + 4 + 5 = (5 \times 6)/2 = 15$$

or that

$$1 + 2 + \ldots + 100 = (100 \times 101)/2 = 5050$$

F. Rectangular Solid Arrays

Inputs: a set S, together with a 1–1 correspondence between the elements of S and the elements of a three-dimensional array, which can be decomposed, for example, into a vertical stack of identical horizontal rectangular arrays. Each such rectangular array, or *layer*, has the same number of rows and columns as every other layer in the stack. For example, if S has eight elements, we could instantiate this frame by arranging the elements of S on the vertices of a cube, and we could view these vertices as forming a vertical stack of two layers, each a two-by-two rectangular array, given by the top and bottom faces of the cube.

FIGURE 2.6. A rectangular array for dealing to three players.

A rectangular solid array can have any number of rows, columns, or layers. Observe, however, that a given rectangular solid array can be broken into parallel layers of rectangular arrays in *three* different ways. For us, these *three* ways are included in the rectangular solid array frame, and we give them equal status. In this sense, we view a rectangular solid array as independent of any particular choice of how to view it as a stack. In particular, we used only the words "vertical" and "layer" above in order to explain the basic concept, by describing how a rectangular solid array might be built of smaller parts. Two alternative stackings, similar in spirit but different in effect, are equally available. We do not favor any of these three stackings, not even the one we used originally, in order to invoke the rectangular solid array frame.

The rectangular solid array subframe, for example, helps to connect three-factor multiplication with volume and so helps explain why whole number multiplication is associative.

G. Dealing

More than the previous ones, this subframe represents a process, unfolding over time, rather than a static, finished structure. We deal the elements of a set S to a certain set of players.[16] The result can be pictured as a rectangular array of elements of S, with one row per player and perhaps several elements of S left over. Each row represents a given player's share of what has, so far, been dealt, and the resulting shares are, by definition, equal. The dealing process continues until there remain too few elements of S to go around to all the players. With three players, the final rectangular array might look like Figure 2.6, and the remaining elements of S would have to number two at most. We form these remaining elements into a second, separate rectangular array, consisting a single row. If S has fewer elements than there are players, the first rectangular array has to be empty, and all the elements of S remain after this "empty deal." On the other hand, if no elements of S remain after the first array is filled, the second array, the row, needs to be empty.

Once a given deal has been completed, we can augment S by adding further members and build on what we have already done[17] in order to complete the dealing of the enlarged, new S.

The dealing subframe connects to whole number division.

Default Evaluations

For the counting construction frame, assigning a default seems, at least at first glance, more like choosing a strong bias than attempting to say something cogent, based on past experience, about what might be typical. Perhaps a more effective default assumption here might be more like a suggestion to explore the input data carefully for one of a range of possible patterns—that is, for possible instantiations of subframes such as those we have described.[18] Each subframe, as we try to match it to data at hand, might have its own defaults, but here, at a finer level of detail, such defaults, if *very* loosely specified, may seem less like biases. This looseness, implemented differently at two distinct but interacting levels of detail,[19] might help explain the difficulty many people have when passing from concrete problems to useful arithmetic.

As an example of the more concrete default that might appear at the level of subframes, that is, specific counting constructions, consider the rectangular array subframe. One potential candidate for a default might be a single row of five, a 1×5 rectangular array, in correspondence with the fingers of one hand. Another likely candidate might be a 2×5 array, which might set the fingers of both hands in correspondence with the numbers 1 through 10.

THE BLOCKS FRAME

Here we represent the physical materials called "multibase arithmetic blocks," also known as MAB or Dienes blocks. This frame, too, is a collage. We use a terminology proposed by our students. This terminology differs from that urged, for example, by the Cuisenaire Company's manual. We discuss this terminology below. Much of what we know about these blocks is folklore, based on many years of use. Discussions of the terminology are indispensable for anyone who works with blocks. Indeed, the things we say about these blocks will matter just as much as what we do with them.

Inputs: a *base* $b > 1$; a *unit*, by which we mean a particular "real block," from the set described below; and a *value*, a given number, with which the unit is equipped.

A. Real Blocks

The most important thing about real blocks is that they are physically present. They are made of wood. We can hold them, move them, build things with them. They are concrete mathematical objects: concrete in the sense that

they are physical and present, mathematical in the sense that they have a structure that allows them to function both as representations for concrete problem situations and as objects to be studied in their own right. Because the blocks are concrete and yet can also serve so readily as symbols, they function as bridges between different conceptual domains, or worlds. We can use them, subject to our choice, to pass between the world of shapes, textures, and weights and the contrasting world of counting, number constructs, and arithmetic. Because we choose (and later, choose again) their meanings, blocks are multivalent, multivocal, and ambiguous at the same time that they are concrete, physical, and precise.

Real blocks come in four kinds, in order of increasing size: *piggies*,[20] *longs*, *flats*, and *solids* (Figure 2.7). Since these blocks are widely used and their terminology, in many variants, is part of the vernacular of classroom practice, we will describe them in a more mathematical language.

The most important single feature of these blocks might be that larger blocks can be built, by hand, for instance, from smaller blocks—and often, most importantly, this can done in different ways. The piggy is a 1-cm cube. A long can be built from, and hence consists of, b piggies. A flat can be built from, and hence consists of, either b^2 piggies, b longs, or some combination of the two. A solid can be built from, and hence consists of, either b^3 piggies, b^2 longs, b flats, or some combination of all three.

So far, the blocks reflect the base alone, which determines their relative sizes. Now we explore the function of the unit, together with its value. Here again, it

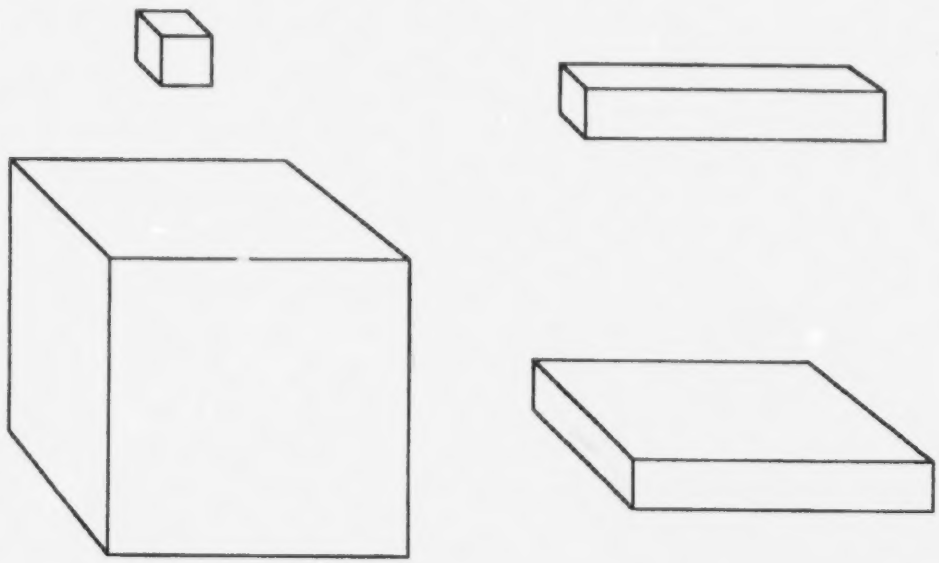

FIGURE 2.7. The blocks. Clockwise from top, left: piggy, long, flat, and solid.

helps that there is no unique or "most natural" choice, either for the unit or for its value. Each assignment of a unit and a value triggers a number representation. For example, if we make the piggy our unit, with the value 1, then the long will represent the number b, the flat will represent b^2, and so forth. In this way, any whole number can be represented (in more than one way, generally) by a collection of blocks. If, however, we take the solid for our unit and give *it* the value 1, then a flat represents $1/b$, the long represents $1/b^2$, and the piggy represents $1/b^3$. This second choice allows us to represent a certain range of fractions in addition to whole numbers.

Whatever the choice of base, unit, and value, any collection of blocks[21] can be *reduced*, in the sense that groups of smaller blocks can be traded for larger blocks having the same total volume, until a collection with the smallest number of blocks, equal in volume (or total number of piggies) to the original collection, can be obtained. This reduced collection is unique.

Most typically,[22] the base b is 10 and the unit is a piggy, with the value 1.[23] Then the number representation provided by the blocks connects naturally with the base 10 place value number system, to represent whole numbers into the thousands. This block system becomes awkward beyond about 10,000 and cannot be used to represent fractions.

For fractions, continuing with $b = 10$, we can make the unit a solid instead of a piggy, with the solid equipped with the value 1. With this choice, a piggy represents .001; and, with this understanding, we can use real blocks to represent decimal fractions down to thousandths.

In the classroom task that we study, the blocks frame enters in the statement of the task, instantiated with $b = 5$, with a piggy as the unit, with the value 1.

B. Imaginary Blocks

Using our imaginations, we can extend the block system so as to represent both very large and very small numbers in base b. Imaginary blocks might be described as "less concrete mathematical objects." We now explain them quite informally.

For large numbers, for example, we can assemble *big longs*, *big flats*, and *big solids*. A big long consists of b solids, lined up in a row and glued together. In the same vein, a big flat consists of b big longs, glued into a square array, and a big solid consists of b big flats glued into a cube. Hence, a big solid consists of b^3 real solids and thus consists of b^9 real piggies. Then b big solids form a *big big long*, and so forth. For small numbers, we subdivide a piggy into b *little flats*, each little flat into b *little longs*, and each little long into b *little piggies*, and so on.

Again, the choice of a unit triggers a number representation, using reduced block collections, but this representation now works smoothly to provide block interpretations for all b-adic decimals, finite or infinite.

C. Block Arithmetic

Once a base and unit are given, we can define arithmetic with blocks, using blocks alone, without reference to a number system. The key idea is to invoke key counting constructions and represent each arithmetic operation by means of a suitable block construction. We begin with a standard, widely used interpretation.

Example 1

We take the base b to be five, the unit to be a piggy, and its value to be 1. Here is one approach to multiplication. For concreteness, we multiply the numbers A and B, given in blocks as follows:

A = one long and three piggies

B = three longs and two piggies

The first step is to build a rectangular array of piggies, whose rows are A piggies long and whose columns are B piggies long. The product A × B is *defined* to be the number of piggies in this rectangular array. If we then organize this rectangular array further, as in Figure 2.8, we can easily find a block representation for A × B and then a reduced representation, if we like, by combining similar blocks.

In this system, a long times a flat, for example, can be represented by a solid, and we can invoke the rectangular solid array frame for a suitable model to explore why, in terms of the definition of the product given above, this must be so.

In this last example, we used the piggy to provide a unit of length (1 cm) and also as a measure of area (1 cm^2).

For a second example, we continue in base 5, but this time we take a different unit. The resulting system will be very different.

Example 2

In base 5, we take the *flat* as unit, with the value 1. In this context, we explore the product of the fractions that we would normally represent as 3/5 and 4/5. Here,

FIGURE 2.8. The rectangular array for A × B.

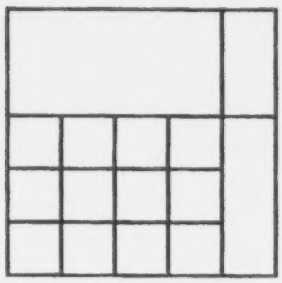

FIGURE 2.9. Rectangular representation of $3/5 \times 4/5$, using a base 5 flat with value 1.

we can represent them by A and B, respectively. In Figure 2.9, we emphasize a rectangular array of piggies at the lower left that is three piggies high, and four piggies wide.

This rectangular array represents the product AB, which we may write as 12/25, because in base 5, each piggy gives $1/5^2 = 1/25$ of a flat, which is our unit, with the value 1. Here, as above, we have used the flat to measure both length (in units of 5 cm) and area (in units of 25 cm^2).

Our third example illustrates the power of imaginary blocks, based on the dealing frame.

Example 3

Again in base 5, this time we take the solid as our unit, with value 1, and attempt to represent 1/3. To do this, we can represent division as dealing, here to three players. We break the solid into five flats and deal these to three players. Each player now has one flat. The dealer holds two flats, which she exchanges for 10 longs, and then she deals three of these longs to each player. Now each player holds one flat and three longs, while the dealer holds one long. She trades this long for five piggies and deals one piggy to each player. Now each player holds one flat, three longs, and a piggy. The dealer has two piggies left undealt. So far, using real blocks, each player holds a quotient of one flat, three longs, and one piggy, while the dealer holds two piggies, the remainder.[24]

Real blocks cannot take us further, so we need to change the rules. The dealer trades each of her two piggies for five *little flats* and deals three of her 10 little flats to each of the three players, leaving one. She trades the one little flat for five little longs, deals one to each player, leaving two, and continues. Each player receives, digit by digit, a block representation of the base 5 expansion of 1/3, an infinitely repeating decimal.

Note here one important point: When we made the step to "little blocks," the little piggy became our "counter," the basic object we were dealing. So the original

set S, viewed as an amalgam of piggies, has quietly transformed itself into a set of *little* piggies, amalgamated differently. And so on, into tinier and tinier blocks.

Each of these examples describes one possible interpretation of the blocks, and each such interpretation enables the blocks to serve as physical embodiments, as metaphoric bricks for building arithmetic structures. Each such interpretation triggers the construction of a special frame, to take its place in someone's blocks collage. As our examples show, related frames can overlap and sometimes interpenetrate. For us, much of the power and fascination of the blocks, as symbols, emerges from the indeterminate range of choices—that is, meanings—they elicit in different settings and at different times. On the one hand, the blocks are very concrete things; but on the other hand, like groups or rings,[25] their connections seem surprisingly diverse.

THE NUMBER SYSTEM FRAME

This frame, too, is a collage. Its assembly might begin[26] with simple chants and then proceed to base 10 whole numbers; base 10 rationals and decimals; and Roman, Mayan, African, Babylonian, Chinese, Inuit, or Polynesian systems, as well as others, perhaps created by the owner and some friends. Here we concentrate on chants and then whole numbers in place value systems—not to favor these unduly, but to illustrate, through the special ways they interact, one corner of a range of possibilities.

A. Chants

Inputs: a set S; also a readily repeatable sequence of words.

Here we pronounce a sequence of standard words, which will let us count as far, at least, as the words available will take us. The counting takes place by matching words in the sequence, one after the other, to objects in S. Here is a simple example:

> This little piggy went to market,
> this little piggy stayed home.
> This little piggy had roast beef,
> this little piggy had none.
> And this little piggy went "Wee, wee, wee, wee!"
> all the way home.

The rhyme scheme and the cadence seem to group the first four piggies in two pairs, perhaps sequentially or, perhaps, linked by the rhymes, in alternation. Against these structured fates, the fifth pig rebels. This rhymed chant is used for counting babies' toes, where the big toe makes a good first pig to go to market, while the little toe, in contrast, seems to represent its owner. Here, as with the grouping of the first four lines, the scheme is multivalent: physically,

the big toe tends to stand alone, while the little toe concludes a group of four. As a counting construction, this rhyme seems especially interesting because it seems to be cyclic.

Ginsburg (1989) describes a child's first numbers as the lyrics of a "number song." We chant "One, two, three, four," and so on, and we learn their pattern as the song unfolds. In particular, we learn the song both while we do—and also while we do not—count objects. Further, we start to count, even before we know much of the song, by touching things and saying names.

B. Place-Value Numbers

Here we restrict to positive whole numbers, for simplicity.

Inputs: a base (a whole number at least 2), denoted b; an ordered set, denoted S, consisting of b symbols; and a *first number*, by definition a particular, designated member of the set of symbols.

We know the base 10 system (with first number 1) well enough, perhaps, to blur our thinking about other possibilities. We give examples in base 5, to illustrate the role of the first number. Both examples appear in the session transcript under study, so we present them carefully to support later analysis.

We are indebted to our students for inspiring the following discussion. The main new feature to observe is how particular counting constructions can emerge directly[27] from the structure (both rhyme scheme and alliteration pattern) of the number chant. Our students first suggested "piggy" as the best name for a basic counter, triggering important cultural associations.

Example 1

Base 5, with symbols O, A, B, C, D, in the given order. Here the first number is O. The chant proceeds as follows.

O	A	B	C	D
OO	OA	OB	OC	OD
AO	AA	AB	AC	AD
BO	BA	BB	BC	BD
CO	CA	CB	CC	CD
DO	DA	DB	DC	DD
OOO	OOA	OOB	OOC	OOD
OAO	OAA	OAB	OAC	OAD
OBO	OBA	...		

and so on, forever. The shorter words always come first, then longer ones.

Example 2

Again base 5, with symbols O, A, B, C, D, in the given order. Here, however, the first number will be A. Now the chant is different.

A	B	C	D	AO
AA	AB	AC	AD	BO
BA	BB	BC	BD	CO
CA	CB	CC	CD	DO
DA	DB	DC	DD	AOO
AOA	AOB	AOC	AOD	AAO
AAA	AAB	AAC	AAD	ABO
ABA	ABB	. . .		

and so on. Again, the shorter words come first.

In these examples, the chant starts with the designated first number, which is different in each case. The successive symbols follow different patterns, too, due to this choice. Nonetheless, the rule for finding the next word in the chant is the same in both cases, and it is the same rule that we have used in base 10. Indeed, denote by W a given chant-word. To find the word that follows W, look at the rightmost symbol in W. If that symbol is O, A, B, or C, then we can replace it with the symbol that comes next in the order, and we are clearly done. If that right-most symbol happens to be D, however, we will need another plan. First, we focus on the word, denoted V, which we obtain by removing the right-most symbol, in this case D, from W. Because V is shorter than W, we know that V already came up in the chant. Hence, we know its successor, which we denote s(V). Finally, we assemble the word that follows W, in this case, by writing s(V) first and then writing an O. This, despite the symbols we have chosen, is the familiar rule.

The place value subframe therefore contains, in addition to the chant and the successor rule, implicit opportunities to group the chant words in rectangular arrays. Indeed, in each example, the division of the number chant by lines and verses corresponds directly to the choice of cases in our application of the next-number rule. As these examples show, the resulting rectangular arrays need not be uniquely determined by the choice of base alone, or even congruent.

Example 3[28]

Here we take the standard base 10 system. Just as 5×5 arrays helped organize Example 2, we know that 10×10 arrays help organize the stan-

dard number chant. But we also tend to organize these 10×10 arrays in stacks of 10, and then we group the resulting $10 \times 10 \times 10$ cubes by thousands, too, and so on, as illustrated by the commas in 3,817,584,293. The resulting nested cubical arrays connect directly to the blocks frame, precisely because the blocks frame uses three spatial dimensions. This connection provides awesome possibilities for representing quite large numbers physically.

The explanatory power of a frame consists not only of its presentation of the broad outline of an idea. For us, the overwhelming value of the frame approach stems from the way it highlights particular details, details which trigger deep associations.

THE COUNTING FRAME

This frame is a collage of two interlinked constructions. We think of both of them as forms of counting. The second needs a number system, but the first does not.

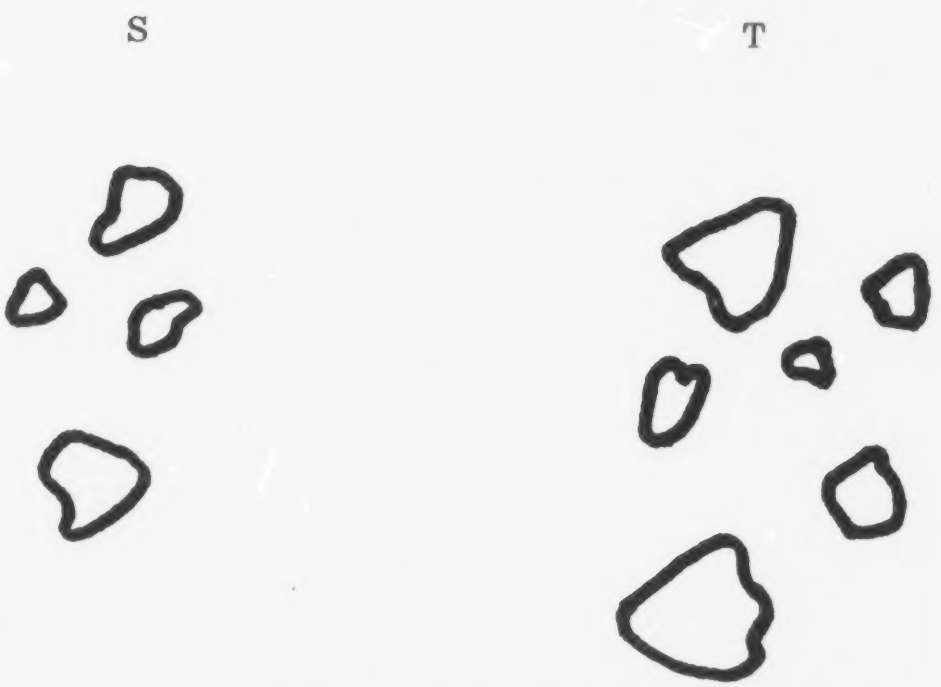

FIGURE 2.10. Two collections of rocks.

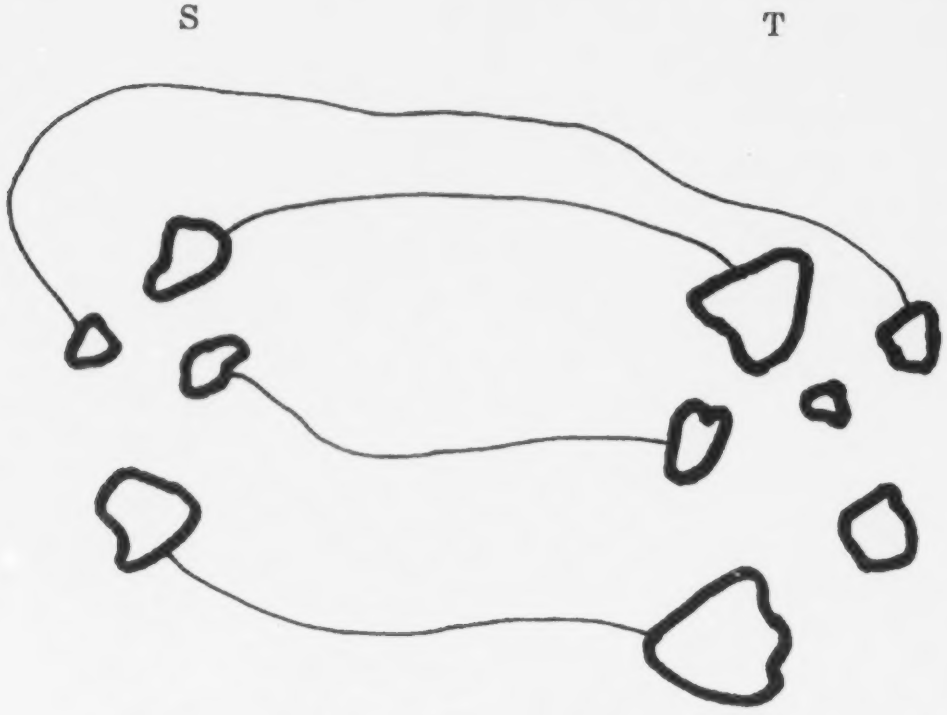

FIGURE 2.11. Strings connecting rocks in S to rocks in T.

A. Matching

Inputs: two sets, S and T. In Figure 2.10, for example, S and T are two collections of rocks.[29]

To illustrate the general process, we now perform a matching on these sets of rocks. For this, we need some string. We begin by physically connecting one rock in S with one rock in T, using a length of string. To continue, we look at the unconnected rocks remaining in S and the unconnected rocks remaining in T. These form subsets S' and T' of S and T, respectively. Either (or both) of these sets may be empty; if so, we are done. If not, we turn to S' and T' and connect one rock in S' to a rock in T'. Then we continue, if possible, with new (and smaller) versions of S' and T', until the process stops. Eventually, all the rocks in at least one of the two sets S and T will have been matched. The result is a 1–1 correspondence between the members of one of our two sets and the members of a subset (maybe the whole set) of the other. In our example, this correspondence is embodied by lengths of string, connecting corresponding rocks (see Figure 2.11).

Here, we see all rocks on the left with strings on them but some rocks on the right without. And this is what we find no matter how we choose—for there are many

choices we can make—to match these sets of rocks with strings. This independence of our choice will be a key ingredient, for it allows us to compare the two sets S and T and say which one is larger, after making only one test, which might be different from someone else's.

If all elements of S and all the elements of T were matched—joined by strings, for instance—we could therefore say that S is *just as big* as T, that S and T have *just as many* things in them, that S and T *have the same size*, or, speaking still more loosely, that S and T are *equal*. Further, no one else could match the rocks differently (if they followed the same rules we followed) and contradict us. In the case above, the sets S and T are clearly not the same size, because some rocks of T remain unmatched, while all the rocks in S are matched. Hence, we say that T is *bigger* than S or has *more rocks* than S does, and again no one can contradict us. Similarly, if rocks in S had been left over, we could then say that S is bigger than T or has more rocks than T does.

Here, we note, no numbers need to intervene. We focus, instead, on matching, how matching gives comparisons, and how each comparison is independent of the particular matching that we used to make it.[30]

B. Counting with Numbers

Inputs: one finite set T; a number system.

Here, returning to our rock paradigm, we replace one set of rocks with the numbers in the given system. For this illustration, we'll take the usual whole numbers in base 10. (We could just as well have used the Babylonian, Chinese, or Maya number system, any number system constructed by the students in our transcript, or any other.) Then, with these two sets, we perform a match (see Figure 2.12).

Here we use the natural order of the numbers (any number system has an ordering, built into its frame) to organize the matches on the left. We start by matching 1, then 2, and so on, until we run out of rocks to match.[31] In this example, the largest number used is 6. Hence, we say that the set T contains *6 rocks*.

Can somebody, however, match consecutive numbers with rocks in T some other way (but matching every rock in T and matching each just once) and then obtain some number of different from 6? If we accept that "same number" or "more than" does not depend on a particular matching, as discussed above, the answer must be "no."

Indeed, suppose that someone else, say, Harry, obtains a count of five rocks in T. If he could do this much, then he could certainly do so with strings, so let us suppose that he has. Harry's strings will lead from his numbers 1 through 5 to all the rocks in T, one string per rock. So, at each rock, let us tie Harry's string to our string leading to the same rock. Now we have long strings, with knots in them, draped over the rocks. The knot-

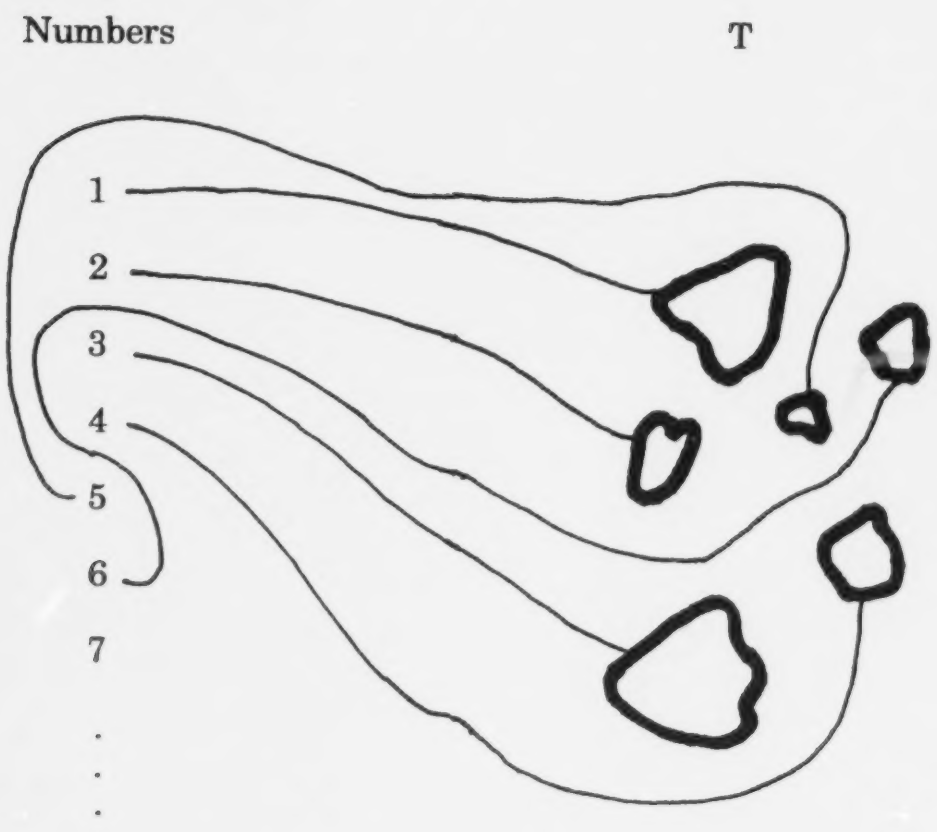

FIGURE 2.12. Connecting rocks to numbers.

ted strings extend from Harry's numbers 1 through 5 to our numbers 1 through 6. These strings match Harry's numbers one-to-one with ours. It follows (because we have a match) that there are just as many numbers from 1 to 5 (Harry's, that is) as there are numbers from 1 to 6 (namely, ours). This is absurd, which justifies our claim.

Here we need to make an important observation. It is just as pertinent to view the numbers here as symbols, representing rocks through matchings, as it is to view the rocks as symbols representing numbers. Often, indeed, we use physical objects, such as rocks or wooden rods, to represent numbers, drawing conclusions from work with the objects and reading these conclusions—validly—as statements about numbers. At the same time—literally in the same context—we may use arithmetic processes, acting solely on numbers, in order to draw conclusions about the physical objects. Which is signifier here, and which is sign? Both sides of a given metaphor may trigger several interpretations, depend on shifting choices, and, only by doing so, make sense.

Default Evaluation

It seems best, once more, to be tentative. As with counting constructions, the default may be a suggestion to explore, in order to instantiate an existing counting construction, or, failing that, to build a match, as with our rocks, one object at a time, perhaps at random. Indeed, counting has its folklore, which may be more determining than the broad perceptual experience that leads to specific counting constructions. Hence, a default for matching and for counting might be the whole numbers 1 through 5, set in correspondence with the fingers of one hand.[32] We have already considered this correspondence, in a slightly different way, as a possible default evaluation for a particular counting construction, the rectangular array subframe. This double choice, at what may be different levels, suggests that if we try to count before we look for patterns, we may be strongly biased toward serial orderings and hence become confused if these do not readily appear. Perhaps here, as in the counting construction frame, the choice of a default may matter less than strategies for what to do when that default has failed to help us.

3

Five Women Build a Number System

A college classroom. Five students sit facing one another on chairs arranged in a small circle. Clockwise from the front are Holly (back to the camera), Katie, Laurel, Mandy, and Marci. Each chair has a widened arm, which forms a small work area. Two other groups of similar size work nearby. Their teacher, Bob Speiser, visits each group occasionally. In the background, Tara Lewis films and Chuck Walter observes.

SCENE 1

Marci: So now, if we get, go into the flats, we need to...I need visuals.
Katie: Rip up a piece of paper. *[Katie gives Marci some paper.]*
Laurel: So when we count, like O, A, B, C, D. And then what's the next?

Here, a *flat* is a five-by-five square array of piggies. Marci places pens and caps, provided by the group, on Holly's open notebook, according to the count. Then she places Katie's sheet of paper on her work area, to her right.

Marci: OO.
Laurel: OO.

Katie: OO?
Marci: OO, OA, OB,OC, OD.
Katie: Wait a minute, wait a minute. What's OO?
Holly: Let's count. Let's count. I need to count.
Laurel: Yeah, I need to know how to count!
Marci: This is OO. This *[joined by others]* would be OO. *[Indicates the pens.]*
Katie: Wait, wait! Stop, stop!

SCENE 2

Marci: Let's count these. *[Gathers pens and pen caps.]* Group: O, A, B, C, D.
Katie: Isn't that D, not O?
Marci: That's D. *[Points to five pens.]*
Mandy: This many is D.
Marci: No, this is D! This is OO.

Marci places a sixth object—a pen cap—to the right of the group of five pens, which she had set in correspondence to the symbol D.

Katie: Oh, oh, *oh*!
[All laugh.]

SCENE 3

Katie: You have a...This is the piggy place. A single in the piggy place...
Holly: So it's actually like this...*[Switches the places of the pens and caps in front of her.]*
Katie: And a single in the long place: OO. *[Displays the hand signal for O on both hands.]*
Marci: *[Adds another pen cap to the piggies pile.]* This is OA.

Katie's transposition has set the group of five pens to the right. Now Marci picks up the sheet of paper that Katie gaven her and prepares to tear it.

SCENE 4

Laurel: Wait! I have a question. Then what is the difference if you're going to call...*[Someone tries to interrupt her.]*
Laurel: Wait, let me talk! If you're going to call this a D, that's a long, too. What are you going to call that?

Laurel leans across the circle toward the group of five pens in Holly's work area.

Laurel: *[Continuing.]* To me it has two different names. It could be zero longs, O longs, or D. What are we going to do?
Marci: We might as well...
Mandy: We don't use long and piggy and flat.
Laurel: To me, if you have the blocks...

Marci begins to tear strips of paper to make counters.

SCENE 5

Katie: Okay, that could be an O long. Couldn't it be a single long?
Laurel: Yeah, it could be.
Marci: It could be an O long, but it's also a D, a D piggy. It has D piggies.
Laurel: So it has two names.
Holly: But it can't have two names, though.
Marci: Yeah, it can.
Mandy: Why can't it have two names?

The symbol D denotes five piggies here. Five piggies, as blocks, constitute a single long, which Katie and Marci refer to, counting longs within their system, as "O long." Hence, two expressions, "D piggies" and "O longs," will represent the same number, given by blocks in two different ways.

Holly: That's going to be too confusing. We've got to decide one way.
Mandy: I wish we could do analogies, because I totally could...
Katie: When you have D piggies, it is the same as an O long. A single long.
Holly: Right.
Katie: Is that any different from saying D, or O long?
Holly: I guess that's the same thing as we did with the blocks. Ten piggies is equal to one long.
Mandy: Mm-hmm.

SCENE 6

Marci: Okay, let's count. *[Continues to tear paper.]*
Mandy: You can't do analogies.
Marci: Okay, let's count!

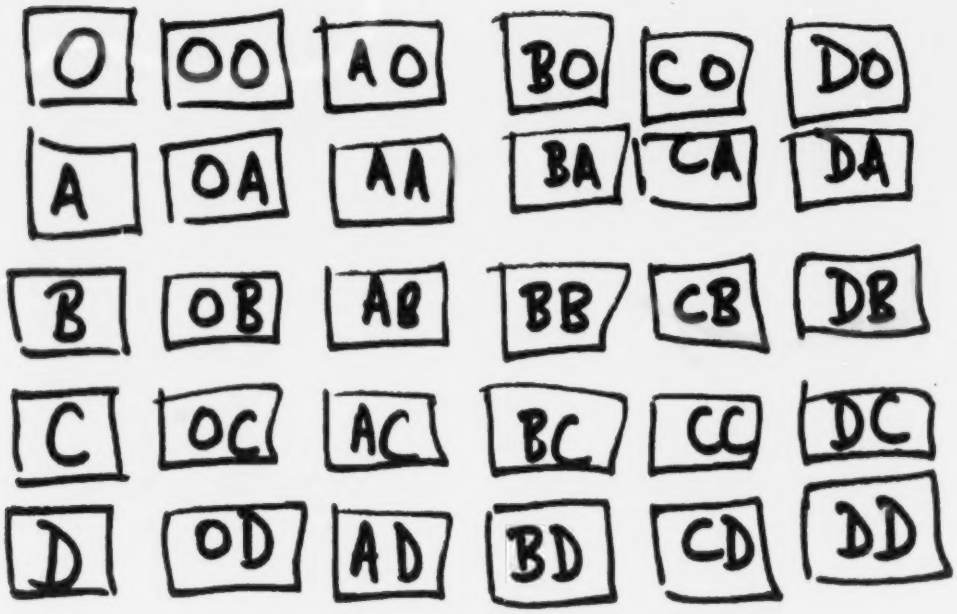

FIGURE 3.1. Marci's first rectangular array.

Marci has torn a sheet of paper to make counters, each roughly a one-inch square, and prepares to place them, one at a time, on another sheet of paper in her work area, beginning with a vertical column.

Mandy: It's place value, it's not really numbers.
Marci: *[Placing counters.]* O, A, B, C, D, *[pauses]* OO.
Mandy: Put it in a different column!
Katie: Take the paper out from underneath!

Marci removes the sheet of paper from beneath the counters and restores their spatial arrangement directly on the surface of her work area, which is beige. The counters, which are white, now stand out clearly.

Mandy: Yeah.
Katie: We're definitely a visual group.
[Others agree.]
Marci: *[Pointing to her first column of counters.]* This is D, OO, OA, OB, OC, OD.

Mandy leans forward, focusing on Marci's placement, as Marci proceeds.

SCENE 7

Katie: OD is the same as A long.

Marci: Now *[placing a counter to begin a third column]* we say, AO…no…yeah, AO.

Mandy: Because we have this many longs. *[Holds up the hand symbol for A.]*

Marci: AO, AA, AB, AC, AD…
[Holly chuckles.]

Marci: BO *[laughter throughout the group*[33]*]*, BA, BB, BC, BD, CO, CA, CB, CC, CD *[starting a new column]* DO, DA, DB.

Katie: And then you just go, when you get into…

Marci: DC, DD. Now…

Mandy: Which is this. This whole flat.

Katie: D flat.[34]

Marci: *[Tears more paper.]* Now we say…

Laurel: 1, 2, 3, 4, 5, 6 *[while Mandy points]*. We have six there. Six by five and we only have five by five here.

Marci's array has six columns of five counters each, in contrast to a flat. Laurel, seated, has been speaking softly.

Marci: Oh, no. Don't say that!

Laurel: That's not the same number. *[Marci laughs, and looks at Holly.]* Whatever.

SCENE 8

Holly: How does that work?

Mandy: How does that work on my paper then?

Marci: Yeah, how did that…

Holly: Because you start with O.

Holly points to Mandy's paper, where Mandy has drawn a five-by-five array of symbols, beginning in the upper left with OO, ending in the lower right with DD.

Holly: *[Continuing.]* You start labeling these OO. *[Points to Marci's* array of paper counters, arrayed on Marci's desk.]

Laurel: *You* start with OO *[points to the upper left corner of Mandy's diagram]* and *we* started with just O *[points to Marci's counter array]*.

Marci: Once we get to here, we've got to be, to an O flat.

Marci indicates the symbol DD at the bottom right-hand corner of Mandy's five-by-five array.

Laurel: Wait! Wait!

SCENE 9

Katie: That's what it is. Lookit[35]: These right here is where you start the multiplication.

Here, Katie has blocked the first column of Marci's counters with her hand, leaving visible the five-by-five array that directly corresponds to Mandy's, which starts, at the upper left, with the compound symbol OO.

Katie: *[Continuing.]* Because when you start multiplying these things you're going to have a...
Mandy: That's true.

Could a compound symbol, like AB or OO, be made to represent a product? Could Mandy's array of compound symbols become—not just resemble—a multiplication table? Another learning group, several feet away, is building a successful system based in part on this idea, just as the conversation here unfolds.

Katie: Wait a minute. An O times an O should be an O, not an OO.
Laurel: O times O should be an O.
Katie: Yeah, an O times an O should be an O.
Holly: Why should an O times an O be an O?
Katie: Because a single times a single is a single.
Holly: Oh *[as Marci laughs]*.
Laurel: But you have an OO which would be in our language...
Katie: In our language...
Laurel: But that's not right...*[Flashes six fingers to Holly.]*
Katie: It would be...*[Also flashes six.]*
Mandy: But if we started with this as...
Marci: Yeah, this is OO. *[Flashes another six.]*
Laurel: But O times O can't be OO.
Marci: But we're not saying it's O times O.
Holly: I think we should start with OO, OA, OB, OC, OD *[pointing to the counters in the first column of Marci's array]*.
Katie: Why?
Laurel: Why?
Mandy: But then it won't work. Because then the number system, the place value system won't work.

SCENE 10

Laurel: Then one piggy is called an OO?
Katie: No, because then you screw up your place values.
Laurel: Yeah.
Marci: Yeah.
Holly: Oh, tha't's right!
Laurel: Oh!
Mandy: Oh!
Katie: Oh!
[They laugh.]

SCENE 11

Katie: Wait a minute. That's just the names of them. Okay. We're not talking multiplication here.
Marci: Yeah, we're not talking multiplication.
Katie: We're just talking about the names.
Laurel: I just want to figure out how to count.
Marci: Exactly. And it's not working now, because we got to here.
Katie: No. Yeah. It is.

SCENE 12

Marci: We got to here, and this is a *flat*.

Marci places her right hand, palm vertical, over the space between the fifth and sixth columns of her array. To the left is a five-by-five square, to the right one column of five counters. With a sweep of her left hand, Marci indicates the left-most five-by-five array, which is not the same as the array Katie emphasized in Scene 9.

Marci: *[Continues.]* But we didn't even start talking about it *[points to* Mandy's paper] until we got [returning to her own last column] to here.
Holly: Well, we could just do it different than she *[Mandy]* has it written on there.
Marci: That's because, I guess, once you get to C, once you get to C you can't say *[pause]* DA, because that would be saying...wait.
Mandy: Wait, the D column is already used.
Marci: Yeah.
Mandy: Because you already multiplied times that. *[Motions to top of the flat.]*

Marci: No, we're not using multiplication. We are trying to count.
Katie: We're trying to just count. This is just your, our number system. This is just what we're calling them.

SCENE 13

Marci: Let's try it.
Holly: Let's add something, and do something with this and see what this looks like.
[Break in the tape.[36]*]*

SCENE 14

Katie: Okay, let me think about this for a second.
Holly: Yeah, I gotta see…
Katie: Okay, so the one's that…You can have A, B, C, D, E, F, whatever, or however many you have. But the one on the very end is the piggies column. I'm just confused. Holly: Just like normal we do.
Marci: Yeah.
Katie: Okay, I was just thinking backwards.
Marci: So, a D…DO would mean you have D of these. *[Motions to Mandy's paper.]* Which D of these, D longs, equals *[points to Mandy's paper again]*, equals O flats ?
Katie: Okay, but when you say DO…
Laurel: This is what confused me from the very beginning, is how we have so many names for one little thing.

Laurel's original objection (in Scene 4, for example) was to writing the symbol D to indicate five piggies, while having no clear symbol for one long, which corresponds to the same number. Now she focuses on D longs, in relation to one flat, and makes a similar objection. The conflict that Laurel emphasizes—between how the blocks accumulate and how the written number symbols change—continues as the represented numbers become larger.

SCENE 15

Katie: Okay, we have a problem. We don't have a nothing. How do you say a single long? No, no, how do you say, how do you say a long?
Marci: Yeah, it's a D.
Laurel: But longs are in this column *[points to Mandy's second place value mark]*. Why are we saying one long…I'm sorry I interrupted you. Keep going with your thought!

Mandy: Yeah, you're right.
Katie: How do you say *a* long?
Holly: I'm now confused. What are you guys trying to get at?

Marci moves to indicate support for Holly's question.

SCENE 16

Katie: How do you say *a* long? Remember? Because if you say AO, that's A long and O piggies. *[Holds up the hand signal for O.]*
Laurel: That's what I was saying.
Marci: Okay, *a* long you could just call "long," or you could just call it "D."
Mandy: We don't need an O, in this. *[Points to her paper, to the block section.]*
Marci: Because you don't really need to know...

The students' energetic experimentation and the speed with which they worked have caught both investigators by surprise. Speiser (S), the designated teacher, joins briefly, hoping for some clues.

SCENE 17

S: *[After watching.]* I see flats. Are they flats?
Laurel: No, they're piggies.
Mandy: Well, almost.
S: These are piggies.
Marci: These are our counting. O, A, B, C, D, OO, OA, OB, OC, OD, AO *[pointing to her counters].*
Holly: I'm really confused.
S: AO?
Katie: It's a flat. It's a long plus a piggy.
Marci: Because...
S: Uh-huh.
Marci: Once, you remember...once you get to D...once you get to D on your longs or here you have...*[motions to columns].*

SCENE 18

S: Wait a minute. What does O represent? Can you show me?
Mandy: It's this. *[Holds up a closed fist, which is the hand symbol for O.]*

Marci: This is O. *[Holds up a single piece of counting paper.]*
Katie: It's a single.
S: Just that.
Marci: Yeah.
Katie: Just a single.
Marci: But if you have OO, this is OO.
Katie: It's a single...
Marci: Because it's a...
Katie: It's a single plus a long, because you've got a single. *[Holds up the O symbol.]* These are your place values. *[Points to the place values on M's paper.]* You've got piggy, long, flat, solid. And so f you have OO, that means you have a single flat and a single piggy. That's OO.
S: A single long plus a single piggy, or a single flat plus a single piggy?
Katie: A single long.
S: A single long.
Mandy: A single flat, plus a single long, plus a single piggy would be OOO.
S: Then would OOO mean...then would OOO mean a flat, a long, and a piggy?
Mandy and Katie: Uh-huh.
S: *[Thinks he begins to understand.]* Okay!
Mandy: It would be a long *[holds up O symbol]*, a flat *[holds up O symbol]*, a piggy *[holds up O symbol]*.

SCENE 19

S: How do you, by the way, represent no piggies at all?
Mandy: We were just working on that.
S: Do you have a decision about that?
Mandy: No, not yet.
S: Or were you just discussing that?
Marci: We just came upon that problem.
S: I'll allow you to continue. This is neat. You guys have...you guys have an interesting system. *[Laughter from the group.]*
S: You really do. Yeah. Keep working. Thank you. *[Exits.]*
Katie: How do you represent nothing?

SCENE 20

Laurel: What I'm thinking about...
Holly: What are you thinking about?
Laurel: Okay, I know I already brought it up. If we have, we have D piggies equal to O long, right?

Marci: No. Yeah.
Katie: Yes.
Marci: Yeah.
Laurel: But then if we write D piggies...*[Points to a diagram of place values on Mandy's paper.]*
Marci: If we just put D right there, then we'll just know...
Mandy: But then you don't, then you don't need a zero. If you put...
Marci: Yeah, you just put...
Laurel: But that's not...D piggies would be...
Katie: D piggies would be...
Marci: It's just like D piggies equals one long...
Holly: In the place value system...
Marci: D longs equals one flat, D flats equals *a* solid.

SCENE 21

Laurel: I understand. So when we say it, we don't have any way...so D...we don't have any way to say...*[Points to Mandy's place values again.]*
Marci: Yeah we do.
Laurel: Okay.
Katie: Say one long.
Marci: You can just say...
Katie: A zero long...
Marci: Just say an O, [37] O long.
Laurel: So if we want to say...
Marci: You could say either way...

SCENE 22

Laurel: Wait, don't interrupt me, please! We have D piggies...that equals a zero long, but we don't know how to say that without saying "long," right? We have to say "zero long."
Katie: O long.
Laurel: Oh, O long.
Mandy: Or you could say D.
Laurel: It's either D or O long. You would have to put "long" in there.
Marci: Mm-hmm.
Laurel: Okay *[uneasily]*. Just, I don't know why I was confused about our place value system.
Holly: I see what you're saying.

SCENE 23

Katie: Well, it's a...See, if we have something to represent "no piggies," that's what we need.

Laurel: Yeah, we really need one. So you could say…D…
Marci: Just say "no piggies."
Katie: But then it'd be dumb to say O…
Mandy: Do we really need it? Do we really need the D? That's my question.
Marci: Yes.
Katie: Yes.
Mandy: Why?
Katie: We do. Because that's…
Holly: *[Slowly.]* Maybe we should try starting with *this* being nothing. *[Holds up the hand signal for O, a closed fist.]*
Laurel: That's what…
Mandy: Because then…
Holly: And see…Then we won't get that confused.
Mandy: Because then as soon as you hit this *[looking at her own place value headings]* right here, then you go to this right over there *[her next heading]* anyway.
Holly: Right.
Marci: And make it an "A long"?
Laurel: Then it would be A.
Katie: But then how would you say…?
Laurel: It would be AO. AO would be one long.
Katie: Lookit, lookit, wouldn't you need to say…
Holly: Oh, goodness!
Laurel: I don't know, that's just what I…
Holly: Maybe it should go…
Katie: Maybe we don't need D's…*[Flashes the hand signal for D: our* fingers.]
Holly: No.
Mandy: We do.

SCENE 24

Holly: No, maybe we should go, like this…
Laurel: I don't think we need to start with this *[holds up the hand signal for O]* representing this *[holds up the hand signal for A]*.

As Marci watches closely, Holly places five counters to form a first column on Marci's work space.

Holly and Marci: A, B, C, D…*[Both stop.]*
Holly: Um…

In the previous array, the first column began with O and ended with D, and the discussion focused on the next entry, corresponding to the first spot in the second column. Here, starting the count with A instead of O, the discussion focuses on the first column's final entry.

Mandy: Then it'd be...
Holly: Then this is an O *[holds up the hand signal for O]*, OO.
Katie: Nothing.
Laurel: Nothing. *[Signals with one hand, as in baseball, to indicate that* she is safe on base.]
Marci: Wait. A...So count: A, B, C, D.
Katie: Then you'd need an O, and that'd be equal to a long.
Holly: Then what would you say?
Laurel: That would be a long.
Laurel: It would be AO.
Marci: You'd just say "long."
Katie: No. Then you'd say AO.
Marci: Then you'd say AO.
Laurel: Right. . .

Marci holds a paper counter in her hand. So far, only the counters corresponding to the symbols A through D have been set down in her array.

SCENE 25

Katie: Put it *[indicates Marci's array]* in the next column. *[Marci obeys.]*
Holly: No! No!

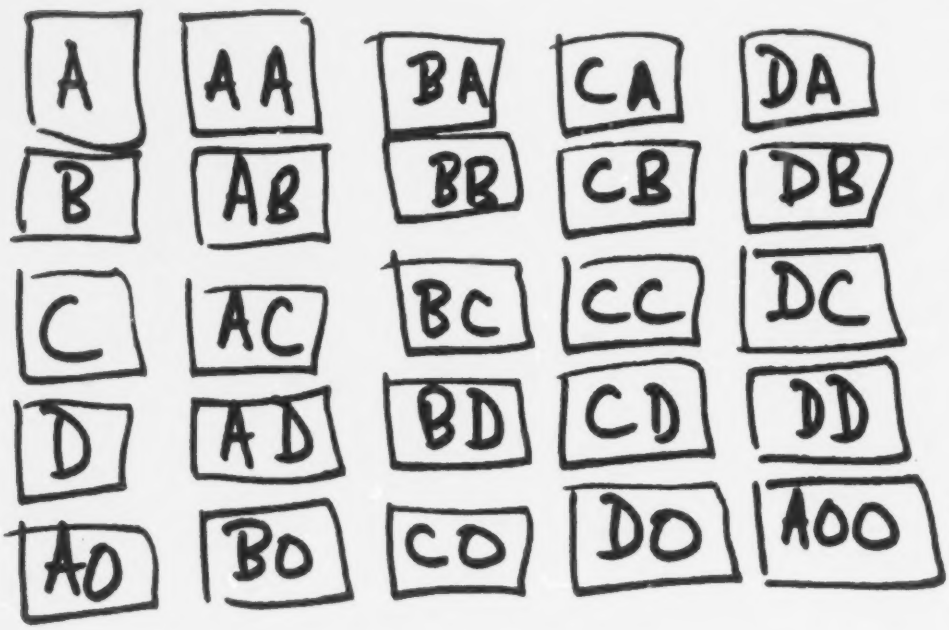

FIGURE 3.2. Marci's second rectangular array.

Mandy: It has to be in *that* column. *[Takes the counter for AO from the top of the second column and puts it at the bottom of the first column.]*
Holly: No, because that has to be AO. You'd have A, B, C, D, AO.
Marci: A, B, C, D, AO. Then how would you go to the next one?
Mandy: Then what would that be? No, that won't work.
Katie: Okay, I get it!
Laurel: It would be AA.
Katie: Then it'd be AA.
Mandy: Oh, yeah.

SCENE 26

Marci: *[Placing further counters in a second column, as Mandy counts in unison.]* AA, AB, AC,...
Katie: No, BO.
Mandy: No, 'cause you're not up that far.
Marci: AA, AB, AC, AD, BO.
Katie: Okay, okay!
Marci: *[Now forming a third column.]* BA, BB, BC, BD, BO.
Katie: No.
Katie, Mandy, and Marci: CO.
Marci: CA, CB, CC, CD, CO.
Mandy: That's cool, 'cause it's in the A, B, and C column. And that's the D column, and that's a flat.
Marci: *[Continues counting.]*...DO.
Laurel: It's just like counting by carrying.
Mandy: And that's a flat.
Holly: Yeah, and that is a flat. *[Motions to the counters].*
Marci: We don't have a flat yet.

SCENE 27

Katie: You've got to go D. You've got to do the D.
Marci: DA, DB, DC, DD. *Then* you say...*[as Laurel joins her]* AOO.
Mandy: Yeah. *[Claps.]*
Marci: So this *[points to the final piggy]* is AOO.
Katie: But it's not *[points to flat on Mandy's paper]* DD.
Mandy: Right. *[Jots something down.]*

Mandy's original array, to represent a flat, began with the symbol OO and ended with the symbol DD.

Holly: Well, her thing is different than what...
Laurel: We're not...
Marci: So now we've got AOO.
Katie: I've got to draw this.

Holly: I've got to draw this, too.
[Break in tape.[38]*]*

SCENE 28

Mandy: I like this system. I think that works good.
Katie: On the bottom you're always changing.

Holly begins to draw a counting chart. Each of the others, save Marci, starts her own within the next few seconds. Marci, however, preserves the counters in her work space. She watches Holly, picks up her open notebook, and then puts it aside.

Holly: This is…wait.
Katie: Okay, that works because a flat is AOO. And a flat…yeah.
Marci: Yeah.
Katie: So then you'd have AOA, AOB, AOC, AOD, AAO.
Mandy: Right.
Marci: Okay…AAO.
Holly: *[With emphasis.]* Whoa!

SCENE 29

Katie: Then AAA *[points to Marci's AAO]*. That would be a flat, a single long, and nothing, no piggies. So then here you've got a flat, a long, and *[Laurel joins in]* one piggy.
Katie: A piggy. AAB, AAC, AAD, and then it would be ABO. *[She* adds this to her chart.]
Mandy: Right.
Holly: And then it'd be a B.
Mandy: Yeah, you just add them. Just like in our…in our place value system, not our number system.
Holly: It's hard because then I think of ten.
Mandy: Because other number systems would be…
Marci: It's hard because we're not...
Holly: And we're not at all…
Laurel: And we're not in it at all. *[Marci agrees.]*
Mandy: You guys, the base systems, the systems with like bases of eight…*[No one seems to hear.]*

SCENE 30

Marci: So how old are you?
Holly: How old are we? I'm sorry, *[to Mandy]*, I totally was like…off with them.
Laurel: CD. I'm CD old.
Mandy: When you're doing—when it's base 8 or base 4, it's the same.

They use place value too.

Marci: I'm D0. I'm DO years old.

Katie: Yeah, me, too. I'm DO, you're CD.

Laurel: Okay.

Holly: What am I?

Marci: How old are you? *[Holly flashes 19 with her hands. Katie laughs.]*

Laurel: I'm CD.

Marci: You're CD.

Laurel: You're CD, too.

Mandy: That's what I am, too.

Holly: Am I CD? How?

Laurel: Let's try counting stuff.

Katie: You've got A, B, C, D, AO *[counting on her fingers].*

Holly: Okay.

SCENE 31

Mandy: So there's AO people in our group.

Marci: AO in our group.

Katie: So, say this group and that group *[points to another student group]* would be BO.

Mandy: No, there's one more person in that group.

Katie: No there's only AO in that group.

Mandy: Someone's not here today.

Katie: Okay, well right now we'll just…Then let's see. *[Looks at a third group.]* In that group they have AA.

Holly: All the others, they have…

Marci: They have AAA.

Holly: They have C…no!?

Marci: They have AA, so it'd be CB. *[Laughs, as others join her.]*

Holly: CA.

Marci: CA…yeah! *[Laughs again.]*

Laurel: For all of us? *[Motions to the entire room.]*

Mandy: Did you count Tara and Dr. Speiser and Dr. Walter?

Holly: No, no.

SCENE 32

Katie: No, wait, I'm confused. Okay, there's AB *[counting the third group].* No, there's AA…But *[as Mandy joins her]* in these two groups there's AO. *[Motions to their group and one* other *group.]*

Marci: *[To Holly.]* I'm DO years old. How exciting! *[Laughs.]*

Laurel: If AA is in that group…*[Points to the third group.]*

Marci: *[Laughing.]* I can't wait to tell my friends about this one. I tell them all the weird things we do in class.
[All laugh.]

4

Analysis

For us, three important elements of how we experience a drama—namely, conflict, suspense, and surprise—reflect specific ways in which we seek to understand the players' actions. These elements follow a *meaningful* progression. Our interpretations of our students' actions center on conflicts in their group, in order to reveal deep changes in the ways these students think and work.[39] By "conflict," we mean open disagreement. Our interpretations build from how we see our students' argument as drama, in particular from how that drama focuses attention.

In the analysis that follows, we approach our students' dialogue from two directions. The first direction is cognitive. We try to clarify *the way each student thinks* about the mathematics she discusses. To do so, we rely on the preceding frames (Chapter 3), which represent a potentially unstable mix of public mathematics (a kind of folklore) and personal construction, in the special context of the course in which the dialogue took place. To see how useful such contingent objects as these frames may be, we test them here against real life.

The second direction is narrative. Our students' actions tell a story, and we try to read that story, just as we might read a play. The play at hand, like other plays, narrates a web of conflicts, disagreements, breaks, and resolutions. It is not enough to capture, in static descriptions, what the action seems to be *about*. We also need to show what *happens*. Hence, our analysis is organized by actions that unfold in time, as conflict, as suspense, as surprise.

Our students' dramas move us when their stories become ours. Each drama represents a change—for individuals, perhaps, but also groups. The overwhelm-

ing data of the moment, such as the cry of a bird, the passage of a car, the conversation at our table, can stay with us only to the extent that we can represent them, frame them, and thus give them meaning. To sense a change depends on what our frames can represent and what, as we go on, might change in them. Our method is to find our students' stories through their disagreements, questions, and experiments—and, in framing these, to intercept their words and actions.

These words and actions start with an insurgency.

INITIAL CONFLICT

The transcript begins as Laurel and Katie, in contrast to the others, pose fundamental questions about the group's number chant and its relation to the blocks. Katie (Scene 1 in Chapter 3) demands a first experiment, which focuses the subsequent discussion. Conflicts rapidly emerge:

> Scene 1:
> Katie: Rip up a piece of paper. *[Katie gives Marci some paper.]*
> Laurel: So when we count, like O, A, B, C, D. And then what's next?

This question seemed to center on the pattern of the number chant, especially its successor rule. But as the chant continues, counting enters, as a distinct, new issue:

> Scene 1, continued:
> Marci: OO.
> Laurel: OO.
> Katie: OO?
> Marci: OO, OA, OB, OC, OD.
> Katie: Wait a minute, wait a minute. What's OO?
> Holly: Let's count. Let's count. I need to count.
> Laurel: Yeah, I need to know how to count!
> Marci: This is OO. This *[joined by others]* would be OO. *[Indicates the pens.]*
> Katie: Wait, wait! Stop, stop!

We are struck by Marci's boldness here: The flat she has begun might well bring down the system, yet she pushes straight ahead.

SUSPENSE

Suspense, for us, denotes clear focus on specific outcomes, seen as "vital signs" to monitor, signs that could indicate potential failure of the enterprise. The enterprise in question here is an energetically conducted experiment designed, at least in

part, to *elicit* that potential failure. Attention centers specifically on the counting constructions that emerge from the group's number system, with particular reference to the way in which those constructions relate to that number system's place values.

Scene 2:
Marci: Let's count these. *[Gathers pens and pen caps.]* Group: O, A, B, C, D.
Katie: Isn't that D, not OO?
Marci: That's D. *[Points to five pens.]*
Mandy: This many is D.
Marci: No, this is D! This is OO.

Marci places a sixth object—a pen cap—to the right of the group of five pens, which she had set in correspondence to the symbol D.

Katie: Oh, oh, *oh*!
[All laugh.]

At this point, Katie appeals to the counting constructions implicit in the base 5 blocks instantiation, in order to emphasize a fundamental issue. She spotlights the symbol D, whose place value forces it to represent five piggies rather than one long:

Scene 5:
Katie: You have a...This is the piggy place. A single in the piggy place...
Holly: So it's actually like this...*[Switches the places of the pens and caps in front of her.]*
Katie: And a single in the long place: OO. *[Displays the hand-signal for O on on both hands.]*
Marci: *[Adds another pen cap to the piggies pile.]* This is OA.

Katie's transposition has set the group of five pens to the right. Next Marci picks up the sheet of paper that Katie had given her and prepares to tear it.

Scene 5:
Katie: When you have D piggies, it is the same as an O long. A single long.
Holly: Right.
Katie: Is that any different from saying D, or O long?
Holly: I guess that's the same thing as we did with the blocks. Ten piggies is equal to one long.
Mandy: Mm-hmm.

The pens provide objects to be counted, one by one, but the *arrangement* of the pens, according to place values, has been driven by the blocks, to which Katie, and then Holly speaking in support, especially, refers. The chant, however, provides its own place values, which seem to shift according to a rhythm of their own.

On the one hand, the counting construction that emerges from the blocks frame, which has organized the pens, starts with a 5×5 rectangular array, whose second column begins with the position corresponding to one long. On the other hand, the counting construction that emerges from the number system frame, as we saw in the last section, begins with a 1×5 array, a single column, whose last entry represents five piggies.

At this point, shifting to paper counters, Marci begins to build the "flat" that the group's *number chant* will trigger. As she does this, each of the others follows the progression of *her own* place values, in order to decide exactly when Marci completes the flat. This choice to reason, as much as possible, inside the framework that each group member made herself, has seemed especially adventurous here, because Marci's experiment, which on the surface seems to be a single act that tests a single construct, will actually provide five tests, of five potentially different constructs. In this student investigation, for once, not only is no "answer" given in advance, but quite possibly as many as *five* answers, drawn from different personal interpretations, could easily emerge.

For each student, indeed, the building of a flat will be a critical test. Katie and Laurel both have raised doubts—to which they will return—about the symbol D, but here the long-term implications, as the chant reaches beyond five piggies or one long (implications that seem much harder to predict), are what the group has chosen to investigate. Attention narrows, therefore, to the flat that will emerge. The minute detail with which the whole group monitors each word, each placement that Marci will make, expressed physically, throughout Marci's construction, by the way each member talks and moves, reflects extreme suspense. If the first column goes down incorrectly, for example, nothing will be learned:

Scene 6:
Marci: Okay, let's count. *[Continues to tear paper.]*
Mandy: You can't do analogies.
Marci: Okay, let's count!

Marci has torn a sheet of paper to make counters, each roughly a one-inch square, and prepares to place them, one at a time, on another sheet of paper in her work area, beginning with a vertical column.

Mandy: It's place value, it's not really numbers.
Marci: *[Placing counters.]* O, A, B, C, D *[pauses]*, OO.
Mandy: Put it in a different column!
Katie: Take the paper out from underneath!

Marci removes the sheet of paper from beneath the counters and restores their spatial arrangement directly on the surface of her work area, which is beige. The counters, which are white, now stand out clearly.

Mandy: Yeah.
Katie: We're definitely a visual group.
[Others agree.]
Marci: *[Pointing to her first column of counters.]* This is D OO, OA, OB, OC, OD.

Mandy leans forward, focusing on Marci's placement, as Marci proceeds.

Scene 7:
Katie: OD is the same as A long.
Marci: Now *[placing a counter to begin a third column]* we say, AO...no...yeah, AO.
Mandy: Because we have this many longs. *[Holds up the hand-symbol for A.]*
Marci: AO, AA, AB, AC, AD...
[Holly chuckles.]
Marci: BO *[laughter throughout the group[40]]*, BA, BB, BC, BD, CO, CA, CB, CC, CD *[starting a new column]*, DO, DA, DB.
Katie: And then you just go, when you get into...
Marci: DC, DD. Now...
Mandy: Which is this. This whole flat.
Katie: D flat.[41]
Marci: *[Tears more paper.]* Now we say...

A surprise is an emergent outcome, from a process too complex to predict. Here, the process is indeed complex, because three frames are now in play: counting construction, number system, and counting—the latter in two instantiations, one for each of two distinct counting constructions. The focused attention, also needed for surprise, on "vital signs," on particular details, those held "in suspense," now spotlights the emerging structure of the flat that Marci builds. To be precise, this structure, as Marci builds it, is a rectangular array with five rows, but the number of its columns is the point in question. This flat, in other words, is *dangerous*.

THE FIRST SURPRISE

It is significant that Laurel counts the columns of Marci's array in the standard base 10 system, because, in doing so, she violates our ground rule to remain within the system that she constructs. Perhaps for that reason she modulates her voice.

Scene 7:
Laurel: 1, 2, 3, 4, 5, 6 *[while Mandy points]*. We have six there. Six by five and we only have five by five here.

Marci's array has six columns of five counters each, in contrast to a flat. Laurel, seated, has been speaking softly.

Marci: Oh, no. Don't say that!

Here we sense a full stop. Marci's engagement, in particular, has not been casual or tentative, given her last response. We have reached a critical event.

MORE CONFLICT

The dialogue (Scenes 8 through 26) that follows provides surprisingly sharp looks into each student's mathematics. The next event begins with Mandy, who was silent through the last exchange. While Marci built the "public flat" with counters, Mandy drew a private one, based on a different symbol sequence. This private flat was five-by-five. For this reason, Mandy does not share the others' sense of fundamental difficulty.

Scene 8:

. . .

Mandy: How does that work on my paper then?

. . .

Holly: *[Continuing.]* You start labeling these OO *[Points to Marci's array of paper counters, arrayed on Marci's desk.]*
Laurel: *You* start with OO *[points to the upper left corner of Mandy's diagram]* and *we* started with just O *[points to Marci's counter array]*.
Marci: Once we get to here, we've got to be, to an O flat.

Marci indicates the symbol DD at the bottom right-hand corner of Mandy's five-by-five array.

Laurel: Wait! Wait!

At this point, the ambiguity of Mandy's symbols (which could, in principle, be read as products) leads Katie to a new reading of Mandy's square array. If Katie's reading makes sense mathematically, then Mandy's construction might, indeed, produce a base 5 flat. On the other hand, Mandy may have read her symbols differently. Are Mandy's symbols multivocal?

Scene 9:
Katie: That's what it is. Lookit[42]: These right here is where you start the multiplication.

Here, Katie has blocked the first column of Marci's counters with her hand, leaving visible the five-by-five array that directly corresponded to Mandy's, which started, at the upper left, with the compound symbol OO.

Katie: *[Continuing.]* Because when you start multiplying these things you're going to have a...
Mandy: That's true.

Could a compound symbol, like AB or OO, be made to represent a product? Could Mandy's array of compound symbols become—not just resemble—a multiplication table? Another learning group, several feet away, is building a successful system based in part on this idea, just as the conversation here unfolds.

Katie: Wait a minute. An O times an O should be an O, not an OO.
Laurel: O times O should be an O.
Katie: Yeah, an O times an O should be an O.
Holly: Why should an O times an O be an O?
Katie: Because a single times a single is a single.
Holly: Oh *[as Marci laughs]*.
Laurel: But you have an OO which would be in our language...
Katie: In our language...
Laurel: But that's not right...*[Flashes six fingers to Holly.]*
Katie: It would be...*[Also flashes six.]*
Mandy: But if we started with this as...
Marci: Yeah, this is OO *[Flashes another six.]*
Laurel: But O times O can't be OO.
Marci: But we're not saying it's O times O.
Holly: I think we should start with OO, OA, OB, OC, OD *[pointing to the counters in the first column of Marci's array]*.
Katie: Why?
Laurel: Why?
Mandy: But then it won't work. Because then the number system, the place value system won't work.

While Mandy explores Katie's new reading, Katie, herself, working details through, finds difficulties. Hence Mandy abandons both her representation and its five-by-five array. Now, every member of the group has recognized the failure of Marci's experiment, each in her own way.

With these recognitions, each one personal and drastic, a threshold has been crossed—a point of no return, a breach. We shall use a new term, *liminality*, to describe the conflicts that emerge, now that the breach has been acknowledged, with its implied demand for restitution or redress.[43] The next few interchanges, where we might hope (in vain) for signs of progress, emphasize the difficult, unclear terrain the group negotiates.

With number symbols and their meanings now in doubt, how might the *correct* insights gained so far be seen as possibilities for further progress? The group begins (Scene 8) with an analysis of symbols, leading to the following exchange, which seems to mark the recognition of a total downfall while also signaling a sense, by every member of the group, of liberation and relief.

Scene 10:
Laurel: Then one piggy is called an OO?
Katie: No, because then you screw up your place values.
Laurel: Yeah.
Marci: Yeah.
Holly: Oh, that's right!
Laurel: Oh!
Mandy: Oh!
Katie: Oh!
[They laugh.]

MORE SUSPENSE

Next, in stages, Katie will transform what began, for her and Laurel, as a negative critique, into a positive proposal. Because proposals can be tested, attention focuses again but is now reframed. Our past experience—Marci's experiment—has shown us how the frame instantiations here, because they are so deeply interlinked, pack powerful surprises. Suspense, again, is evidenced by focusing on fine details, as Katie, joined by Laurel and Holly in turn, develops what might become a central insight. Perhaps most striking in the conversations we will now examine, is the frequency, and the intensity, of expressions of uncertainty and doubt. In Scene 14, the group continues to explore.

Katie: Okay, let me think about this for a second:
Holly: Yeah, I gotta see…
Katie: Okay, so the one's that...You can have A, B, C, D, E, F, whatever, or however many you have. But the one on the very end is the piggies column. I'm just confused.
Holly: Just like normal we do.
Marci: Yeah.
Katie: Okay, I was just thinking backwards.
Marci: So, a D…DO would mean you have D of these. *[Motions to Mandy's paper.]* Which D of these, D longs, equals *[points to Mandy's paper again]*, equals O flats?

Marci seems to take up the idea of a product notation here and hence interprets the symbol DO as representing D objects of the form O. This raises a startling possibility. Consider the phrase "O long" for a single long, which we read in precisely the same way. To build a number system where the single long is symbolized in place of five piggies, wouldn't the symbol O (or whatever represents a single piggy) need to read multiplicatively, in the sense that it tells *how many* longs to work with? There is, indeed, a multiplicative structure built into place value numbers, captured in the nested regular arrays of their implicit counting construction.

From this viewpoint, Marci's question may ask for precisely *what*, in this case, needs to be multiplied. Katie, who rejected the symbol OO as a first number, because the symbol O already represents a piggy, speaks next.

Katie: Okay, but when you say DO...
Laurel: This is what confused me from the very beginning, is how we have so many names for one little thing.

Laurel's objection, too, drives deep into the meaning of the number symbols. Did the troublesome five-by-six flat emerge *because* the symbol D, in the first column, represented five piggies? Laurel seems to focus on the ambiguity that the symbol "D," for five piggies, also represents "O long." Again, we hear an echo of

the theme sounded by Marci's question: What, exactly, needs to be multiplied? Now Katie makes a key connection, by shifting from D piggies (which focuses on multiplication by five) to one long (which focuses on multiplication by one). In taking up one long, she has implicitly removed five piggies. Now, how might we represent their absence?

Scene 15:
Katie: Okay, we have a problem. We don't have a nothing. How do you say a single long? No, no, how do you say, how do you say a long?
Marci: Yeah, it's a D.
Laurel: But longs are in this column *[points to Mandy's second place value mark]*. Why are we saying one long…I'm sorry I interrupted you. Keep going with your thought!
Mandy: Yeah, you're right.
Katie: How do you say *a* long?
Holly: I'm now confused. What are you guys trying to get at?

Marci moved to indicate support for Holly's question.

This detail—Marci's nonverbal assent—suggests that her earlier question was not rhetorical. She still wants to see what Katie means, and her support of Holly suggests that she, too, is confused. Hence, Katie responds.

Scene 16:
Katie: How do you say *a* long? Remember? Because if you say AO, that's A long and O piggies. *[Holds up the hand-signal for O.]*

The more we reflect on this response, the more we wonder. The word transcribed as "a" in Katie's question might just as well denote the symbol "A." In her next sentence, she gives the hand symbol for O—a closed fist—and shows just one of these. In the group's place value system, whatever number "A" may represent for Katie, here it tells how many longs. But O, as Katie stresses, means one piggy. Hence, the problem with "AO," however many longs the symbol "A" may represent, comes from the *piggy*. That piggy, here, prevents the representation of one long by means of a symbol that counts *longs*. Again, the theme of multiplicity emerges, to support Katie's central assertion (Scene 15) that "We don't have a nothing."

Scene 16, continued:
Laurel: That's what I was saying.
Marci: Okay, *a* long you could just call "long" or you could just call it "D".
Mandy: We don't need an O, in this. *[Points to her paper, to the block section.]*
Marci: Because you don't really need to know...

At this point, Speiser visited, asked several questions, and withdrew as gently as he could. Immediately (the end of Scene 19) Katie restates her earlier proposal (Scene 15) as a question.

Katie: How do you represent nothing?

This comment will elicit further conflict. Laurel, who had expressed her hesitation, raised her doubts, and received little, so far, in direct response, breaks in first.

Scene 20:
Laurel: What I'm thinking about...
Holly: What are you thinking about?
Laurel: Okay, I know I already brought it up. If we have, we have D piggies equal to O long, right?
Marci: No. Yeah.
Katie: Yes.
Marci: Yeah.
Laurel: But then if we write D piggies...*[Points to a diagram of place values on Mandy's paper.]*

Laurel again, speaking intensely from the margin, spotlights the problematic symbol D. If D, indeed, remains constrained to represent five piggies, then how could Katie's plan succeed? The breach Katie began, by shifting from D piggies to one long, has widened. With D abolished, what might play its central role of structuring the number chant in groups of five?

Scene 20, continued:
Marci: If we just put D right there, then we'll just know...
Mandy: But then you don't, then you don't need a zero. If you put...
Marci: Yeah, you just put....
Laurel: But that's not...D piggies would be...
Katie: D piggies would be...
Marci: It's just like D piggies equals one long...
Holly: In the place value system...
Marci: D longs equals one flat, D flats equals *a* solid.

Scene 21:
Laurel: I understand. So when we say it, we don't have any way...so D...we don't have any way to say...*[Points to Mandy's place values again.]*
Marci: Yeah we do.
Laurel: Okay.
Katie: Say one long.
Marci: You can just say...
Katie: A zero long...
Marci: Just say an O, O long.
Laurel: So if we want to say...
Marci: You could say either way...

Now Laurel cannot be stopped. The old argument that D piggies represent one long, which Marci, and perhaps Mandy, now maintains, needs to be dealt with. Laurel refocuses attention on Mandy's columns for place values, to emphasize that shifting from D piggies to O longs shifts columns. Again, she counts outside the system, but this time only for a moment.

Scene 22:
Laurel: Wait, don't interrupt me, please! We have D piggies...that equals a zero long, but we don't know how to say that without saying "long" right? We have to say "zero long".
Katie: O long.
Laurel: Oh, O long.
Mandy: Or you could say D.
Laurel: It's either D or O long. You would have to put "long" in there.
Marci: Mm-hum.
Laurel: Okay *[uneasily]*. Just, I don't know why I was confused about our place value system.
Holly: I see what you're saying.

Holly's comments have been sparse up to this point, but nearly all of them reflect a clear concern with precisely how place value changes correspond to column shifts. Her support for Laurel now suggests that she, too, sees the objections to the symbol D as serious. So far, however, Holly has been silent about Katie's insistence that the system needs to represent a "nothing." Now Katie, stage center, pulls her oar. Will any of the others join her?

Scene 23:
Katie: Well, it's a...See, if we have something to represent "no piggies," that's what we need.
Laurel: Yeah, we really need one. So you could say...D...
Marci: Just say "no piggies."
Katie: But then it'd be dumb to say O...
Mandy: Do we really need it? Do we really need the D? That's my question.
Marci: Yes.
Katie: Yes.
Mandy: Why?
Katie: We do. Because that's...

Mandy's question about needing D seems multivocal. Marci and Katie may well interpret its intent in different ways. Marci still needs D to represent five piggies in her number chant. But Katie needs this symbol too, but for a completely different reason. The symbol D, as given by the task design, needs to remain. The central question for Katie, therefore, based on her comments, has to be its meaning. But to decide what D should mean, as Katie knows, depends on

what we mean by O. Implicitly, the central question narrows to the meaning of the symbol O.

We pause to comment on the underlying mathematics. At this point, the group's instantiation of the number system frame has come heavily into question. Nonetheless, its base remains set at 5, via the blocks, and its symbol order, O, A, B, C, D, had not been challenged. But the *first number*, originally O, has just been made the focus of attention. It seems that this particular default assumption, which may seem inconsequential at first glance, has unforeseen impact. This single choice, whose consequences have already wrecked the enterprise, needs to be made again.

Now Holly puts her oar into the water.

Scene 23, continued:
Holly: *[Slowly.]* Maybe we should try starting with *this* being nothing. *[Holds up the hand-signal for O, a closed fist.]*
Laurel: That's what...
Mandy: Because then...
Holly: And see...Then we won't get that confused.
Mandy: Because then as soon as you hit this *[looking at her own place value headings]* right here, then you go to this right over there *[her next heading]* anyway.
Holly: Right.
Marci: And make it an "A long"?
Laurel: Then it would be A.

Now, according to the symbol order that remains in force, we have "A" long, with A representing one, as Katie, earlier, may well have meant. If Laurel's strong support for Katie's negative critique opened a second breach, then Holly's reconstruction of the symbol O, which assigns to A the role of the first number, not only validates that breach, but it also seems to give coherent form to Katie's positive proposal. The next key step, significantly by Marci, passes literally from zero to one, and hence begins a number chant. That chant, in large part, will rebuild the number system frame. To what extent, however, does each of the others grasp this implication?

Scene 23, continued:
Katie: But then how would you say...?
Laurel: It would be AO. AO would be one long.
Katie: Lookit, lookit, wouldn't you need to say...
Holly: Oh, goodness!
Laurel: I don't know, that's just what I...
Holly: Maybe it should go...
Katie: Maybe we don't need D's...*[Flashes the hand signal for D: four fingers.]*
Holly: No.
Mandy: We do.

Did Holly's proposal catch Katie by surprise? Perhaps. Here again, both the hesitations and the affirmations suggest complexly woven personal reactions. Now, without even a moment to unpack this last exchange, the group begins, with Marci in the lead, to build a second flat.

FURTHER SURPRISES

First, as everyone checks carefully, the symbol AO goes to the *bottom* of the first column.

Scene 24:
Holly: No, maybe we should go, like this...
Laurel: I don't think we need to start with this *[holds up the hand signal for O]* representing this *[holds up the hand signal for A]*.

As Marci watches closely, Holly places five counters to form a first column on Marci's work space.

Holly and Marci: A, B, C, D...*[Both stop.]*
Holly: Um...

In the previous array, the first column began with O and ended with D, and the discussion focused on the next entry, corresponding to the first spot in the second column. Here, starting the count with A instead of O, the discussion focused, instead, on the first column's final entry.

Mandy: Then it'd be...
Holly: Then this is an O *[holds up the hand signal for O]*, OO.
Katie: Nothing.
Laurel: Nothing. *[Signals with one hand, as in baseball, to indicate that she is safe on base.]*

The first discovery is that the flat cannot begin with O, which now counts nothing. In the symbol sequence, A comes next, and so its meaning seems to be determined.

Scene 24, continued:
Marci: Wait. A...So count: A, B, C, D.
Katie: Then you'd need an O, and that'd be equal to a long.
Holly: Then what would you say?
Laurel: That would be a long.
Laurel: It would be AO.
Marci: You'd just say "long."
Katie: No. Then you'd say AO.
Marci: Then you'd say AO.
Laurel: Right...

Marci holds a paper counter in her hand. So far, only the counters corresponding to the symbols A through D have been set down in her array.

The placement of the fifth counter, which corresponds to the symbol AO, is critical. So far, the first column of the new array contains four counters. In the previous array, AO, with two digits, began the second column.

Scene 25:
Katie: Put it *[indicates Marci's array]* in the next column.
[Marci obeys.]

Is the column placement dictated by the change in place value? If so, this flat is forced to have four rows.

Scene 25, continued:
Holly: No! No!
Mandy: It has to be in *that* column. *[Takes the counter for AO from the top of the second column and puts it at the bottom of the first column.]*
Holly: No, because that has to be AO. You'd have A, B, C, D, AO.
Marci: A, B, C, D, AO. Then how would you go to the next one?
Mandy: Then what would that be? No, that won't work.

Mandy's place value headings helped to motivate the previous construction and so formed one basis for Laurel's strong critique. The flat under construction, like the number chant to which it corresponds, reflects place value shifts in a new way. Is this why Mandy hesitates? The same event, for Katie, may represent the end of her long search.

Scene 25, continued:
Katie: Okay, I get it!
Laurel: It would be AA.
Katie: Then it'd be AA.
Mandy: Oh, yeah.

Now Mandy, too, appears to be convinced. In this new system, a place value change should end, rather than begin, each column. Further, with this understanding, AOO completes whatever flat the current number system builds. But—this is evidently crucial—what will be this flat's *dimensions*?

Scene 26:
Marci: *[Placing further counters in a second column, as Mandy counts in unison.]*
AA, AB, AC,...
Katie: No, BO.

Is Katie unconsciously jumping ahead?

Scene 26, continued:
Mandy: No, 'cause you're not up that far.
Marci: AA, AB, AC, AD, BO.
Katie: Okay, okay!

Now it seems, Katie, Marci, and Mandy build identical patterns. With the first number and the chant now verified in five-word lines, only completion of the flat remains. Holly and Laurel, silent here, watch closely. As the focus narrows, even after two full years of study, the next exchanges seem riveting.

Scene 26, continued:
Marci: *[Now forming a third column.]* BA, BB, BC, BD, BO.
Katie: No.
Katie, Mandy, and Marci: CO.
Marci: CA, CB, CC, CD, CO.
Mandy: That's cool, 'cause it's in the A, B, and C column. And that's the D column, and that's a flat.
Marci: *[Continues counting.]*:...DO.
Laurel: It's just like counting by carrying.
Mandy: And that's a flat.
Holly: Yeah, and that is a flat. *[Motions to the counters.]*
Marci: We don't have a flat yet.

Here, perhaps, the older meaning of the symbol D, which represented five, led Holly to conclude that five columns were in place. Katie, in the next exchange, may also mean five when she says D, but with a very different intent.

Scene 27:
Katie: You've got to go D. You've got to do the D.
Marci: DA, DB, DC, DD. *Then* you say...*[as Laurel joins her]* AOO.
Mandy: Yeah. *[Claps.]*
Marci: So this *[points to the final piggy]* is AOO.
Katie: But it's not *[points to flat on Mandy's paper]* DD.
Mandy: Right. *[Jots something down.]*

Mandy's original array, to represent a flat, began with the symbol OO and ended with the symbol DD.

Holly: Well, her thing is different than what...
Laurel: We're not...
Marci: So now we've got AOO.
Katie: I've got to draw this.
Holly: I've got to draw this, too.

As Holly joins in, Laurel, silent through the last exchanges, speaks now, it seems, only to emphasize that Mandy's old array has been replaced. That Katie and Holly, who have argued formally, in Piaget's sense of thinking about actions, begin to draw, suggests not only that they feel they have completed building something, but also that what they have just completed building is, in an important sense, visual. At once it represents a base 5 flat and a five-by-five array. As Katie and Holly draw, every member of the group, save Laurel, will continue to reflect.

Scene 28:
Mandy: I like this system. I think that works good.
Katie: On the bottom you're always changing.

Holly begins to draw a counting chart. Each of the others, save Marci, starts her own within the next few seconds. Marci, however, preserves the counters in her work space. She watches Holly, picks up her open notebook, and then puts it aside.

Holly: This is...wait.
Katie: Okay, that works because a flat is AOO. And a flat...yeah.
Marci: Yeah.
Katie: So then you'd have AOA, AOB, AOC, AOD, AAO.
Mandy: Right.
Marci: Okay...AAO.
Holly: *[With emphasis.]* Whoa!

The completion here, this time on paper, *still* seems a surprise. This flat's success, perhaps no less surprising than the last flat's failure, seems to complete the number system frame, in such a way as to connect, through its counting constructions, to the blocks instantiation. Now, as Katie leads, Laurel, who had fought so hard to make a central point, breaks her silence to join Katie as she chants.

Scene 29:
Katie: Then AAA *[points to Marci's AAO]*. That would be a flat, a single long, and nothing, no piggies. So then here you've got a flat, a long, and *[Laurel joins in]* one piggy.
Katie: A piggy. AAB, AAC, AAD, and then it would be ABO. *[She adds this to her chart.]*
Mandy: Right.
Holly: And then it'd be a B.
Mandy: Yeah, you just add them. Just like in our...in our place value system, not our number system.

Mandy, here, seems to connect the chant's successor rule to the arithmetic action of adding one repeatedly. This fundamental action, as she now asserts, is independent of the base. For Mandy, in other words, the base is now a terminal

whose value can be reassigned within a number system frame whose basic information, such as the rules for symbol order or for adding one, remains relatively stable.

Scene 29, continued:
Holly: It's hard because then I think of ten.
Mandy: Because other number systems would be...
Marci: It's hard because we're not...
Holly: And we're not at all...
Laurel: And we're not in it at all. *[Marci agrees.]*
Mandy: You guys, the base systems, the systems with like bases of eight... *[No one seems to hear.]*

We sympathize with Laurel here, because the group's new number system has been in play for less than five minutes. Mandy's point, too, despite the others' silence, seems well taken. Given how short a time the group has had to learn its latest system, the *fluent* work the group just did, indeed, needs explanation. Since fluency depends on practice, it must be based on past experience that has remained in place (perhaps a frame) as well as what, specifically, has changed (perhaps its terminal assignments).

FURTHER CHALLENGES

More practice, in this classroom, comes from posing tasks.

Scene 30:
Marci: So how old are you?
Holly: How old are we? I'm sorry *[to Mandy]*, I totally was like...off with them.

The respect that Holly shows for Mandy's feelings here suggests how deeply Holly has been listening, and not just to Mandy. Perhaps this drama can be read, from several directions, as a case study about listening and care.

Scene 30, continued:
Laurel: CD. I'm CD old.
Mandy: When you're doing, when it's base 8 or base 4, it's the same. They use place value too.
Marci: I'm DO. I'm DO years old.
Katie: Yeah, me too. I'm DO, you're CD.
Laurel: Okay.
Holly: What am I?
Marci: How old are you?
[Holly flashes 19 with her hands. Katie laughs.]
Laurel: I'm CD.

Marci: You're CD.
Laurel: You're CD, too.
Mandy: That's what I am, too.
Holly: Am I CD? How?

At the beginning of the story, Laurel said that she wanted simply to know how to count. Here, once more, she asks to count within the system. Just as she joined Katie earlier, this time Katie will join her.

Scene 30, continued:
Laurel: Let's try counting stuff.
Katie: You've got A, B, C, D, AO *[counting on her fingers].*
Holly: Okay.

The next exchange will set a further task, which reconnects the players to the larger classroom scene.

Scene 31:
Mandy: So there's AO people in our group.
Marci: AO in our group.
Katie: So, say this group and that group *[points to another student group]* would be BO.
Mandy: No, there's one more person in that group.
Katie: No, there's only AO in that group.
Mandy: Someone's not here today.
Katie: Okay, well right now we'll just...Then let's see. *[Looks at a third group.]* In that group they have AA.
Holly: All the others, they have...
Marci: They have AAA.
Holly: They have C... no!?
Marci: They have AA, so it'd be CB. *[Laughs, as others join her.]*
Holly: CA.
Marci: CA...yeah! *[Laughs again.]*
Laurel: For all of us? *[Motions to the entire room.]*
Mandy: Did you count Tara and Dr. Speiser and Dr. Walter?
Holly: No, no.

Here Laurel, personally, widens the context to include each person in the room, leading Holly to affirm, with emphasis, that the count so far excludes the research team. Now Katie, as she has done with such effect before, will interrupt the flow.

Scene 32:
Katie: No, wait, I'm confused. Okay, there's AB *[counting the third group].* No, there's AA...But *[as Mandy joins her]* in these two groups there's AO. *[Motions to their group and one other group.]*
Marci: *[To Holly.]* I'm DO years old. How exciting! *[Laughs.]*
Laurel: If AA is in that group...*[Points to the third group.]*

Marci: *[Laughing.]* I can't wait to tell my friends about this one. I tell them all the weird things we do in class.
[All laugh.]

The drama opened with a strong insurgency by Katie and Laurel. As the transcript closes, these two participants again pose questions. Marci, whose bravery and good sense helped to inspire the foregoing analysis, acknowledges, with laughter, what fine stories such questions might initiate.

5

Discussion

Our understanding grew in stages, beginning well before we posed the task we study here. Indeed, the selection of the task itself grew from an initial orientation, which first triggered and then helped focus our analysis. We began with how we saw school mathematics as it played out physically, beginning with the main materials we used—here, base 10 and base 5 blocks. As our students built their arguments, they referred to blocks to anchor their major claims. We view these claims and the discussion that surrounds them as texts in progress that are written, argued over, and revised.

TEXTS

Our purpose in this chapter is to understand the argument that Laurel and Katie won. What all five students said and did in our class can be seen, in part, as a rehearsal for their later work as teachers. From this angle, the argument we study offers windows on how strongly held demands for intellectual rigor can drive not just mathematical constructions—here, of number systems—but also wider inquiries.

We approach Laurel and Katie's discourse, through their words and actions, as a *text*, and we read this text, against the drama that surrounds it, first as an investigation into fundamental mathematics; and then, using this initial reading as an anchor, as part of a critique that several of the students build in later texts of how

teachers and schools have failed to take their learners and their learners' aspirations seriously.

The text we study here[44] was written down only in part, but every member of the group has worked with it in several versions. The changes that it undergoes and how it functions in the larger drama yield important glimpses of our students' understanding not just of the mathematics they are building but also of the dominant discourse (Speiser & Walter, 1997), which the texts they build and argue will, because they comment on that discourse, help us to examine.

For us, roughly, a text is something that we interpret in a social context. More precisely:

> A text, for us, is something that a group of investigators builds, perhaps temporarily, for the purpose of discussion, reconsideration and revision. We may not write it down, even in part, although, in principle, we will tend to think we could. The point is that we keep referring to it, discussing what it says, constructing what it means, inquiring into issues which it raises. (Maher & Speiser, 1998)

As we have written this book, our discussions about text and narrative have led to further work and wider conversations: for example, the joint paper (Speiser & Walter, 1997) about how two fifth-graders discovered an important piece of algebra to help them make a point not very different from the one we are discussing here; and a detailed, book-length study (Maher & Speiser, 2000) of how a group of children, including the Gang of Four, beginning at age six, construct and share important mathematics through 11 years.

IMAGES

We and our students often use the blocks we have described above as concrete mathematical objects (Maher & Speiser, 1997a, 1997b). As such, we have described their properties, independent both of the base and of any number system they might later represent, with special emphasis on how important *information* gained through working with them, might be organized. To do so, it seemed to us most natural to build *frames*, in the sense of Davis (1984), so that key observations in base five can anchor further observations that will hold for any base.

To discuss the symbolic number systems that our students built, argued about, and often modified, we developed further frames, first for number systems and later for what we have called "counting constructions." All our frames (through their so-called terminals or slots) allow substantial variation and adjustment.

In postulating the frames we did, we constrained our working constructs to describe, as precisely as we could, both our students' discourse about blocks and numbers and our own. Our goal, especially, was to locate the particular varia-

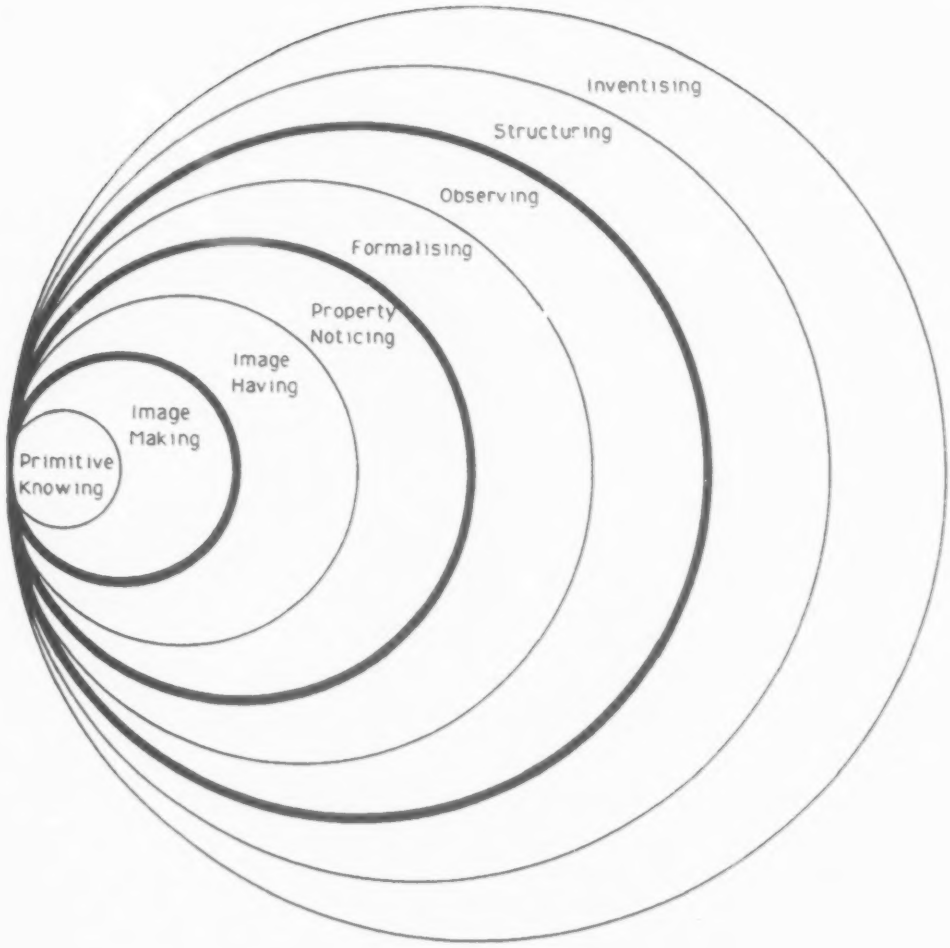

FIGURE 5.1. From Pirie and Kieren (1992, p. 246). Reprinted by permission of the publisher.

tions that we and *our students*[45] chose to work with, within a range we could describe, determined by our frame instantiations. This range, therefore, reflects hypotheses that both we and our students tested, from our different standpoints, through parallel but in many ways distinct investigations. In particular, our students' *images* of blocks provide foundations for the text that they first build, then battle over and revise.

The central images of blocks, for us, are the two rectangular arrays that Marci built with paper counters. Pirie and Kieren (1992) provide a useful framework for the bootstrap process through which concrete images can motivate and test abstractions. As we apply their model to describe key aspects of our students' arguments, we review their terminology. Figure 5.1 shows the main events.

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Image making. In Marci's first array, the paper counters are arranged so that each column represent the five piggies that must constitute a long. Hence the resulting image, expected by some members of the group to give a flat, reflects no further arbitrary choices.

Image having. There ensues a successful verification that the paper array, as it is being built, embodies the counting construction implied by the group's first number system.

Property noticing. The array of paper counters is not square. Hence, this image (from the current number system frame) cannot be isomorphic to a flat. This failure supports Laurel and Katie, who objected, on important grounds, to the group's assignment of the symbol D to represent a long (in the second column) through its five piggies (in the first). Marci and Mandy now attempt minor adjustments, which soon fail. In other words, the images constructed from the group's initial system are abandoned. Without an image to be had, the outer layers of Pirie and Kieren's model cannot possibly be entered.

At this point, however, Katie steps *outside* the system, and hence outside the diagram in Figure 5.1, in order to assert, "We have a problem. We don't have a nothing." Holly responds by raising a closed fist (the signal for the symbol O) and then suggests that this symbol should be redefined to stand for "nothing." This choice engenders a construction.

Folding back. The second paper array is built (image making) and tested for consistency (image having).

> When faced with a question or circumstance at any level...which is not immediately resolvable, we argue that one can *fold back* to any inner level...in order to extend one's current, inadequate understanding. This return to inner level activity is not the same, either in quality or intent, as the earlier, original inner level understanding of that topic. (Pirie & Kieren, 1992, p. 248)

This second array tests Katie's critique, by assigning the symbol O to represent "nothing," and hence leaving five piggies without a special symbol of their own. While Marci's first array was built *within* the system being tested, the second one responds to critiques directed from *outside* that system. Thus, the group's return to image making reflects intents and qualities quite different from the ones that drove the building of the first array.

Property Noticing. The second array confirms Katie's critique by corresponding to a flat and hence supports Laurel's objection to two symbols for a long. Again, the content and the quality are different.

Formalizing. Next, comparing ages, every member of the group begins to formalize, to cross the "don't need" boundary (the solid circle beyond *property noticing*) that surrounds the concrete mathematics that has anchored what has now become symbolic.

In this analysis, we followed the group's arguments about their number system as they built it, tracing the character and use of images which that number system gave them. Key verifications, we have seen, took place beyond the *image-having* level, where arrays of counters were compared to images of blocks. These block images, however, were never problematic; hence, in Pirie and Kieren's model, they must be part of what these authors call *primitive knowing*. In our students' later arguments, they will penetrate the don't need boundary several more times to ground arithmetic with numbers in arithmetic with blocks. The number system, however, in that later context, becomes primitive knowing for these students—no longer mathematics in the making.

In retrospect, the questions we pursued initially seem all too simple. What, precisely, would our students do when pushed beyond rote reliance on base 10 place value numbers and familiar algorithms? How would *they* decide to ground their counting and arithmetic? On what basis, for example, would they define a product or a sum when the numbers they were using were most likely still under intense discussion? And how, precisely, might they represent, and then explain, what they had built?

As we watched our videotape, the unfolding drama caught us by surprise. We did not expect to see fierce argument about what number symbols ought to mean, nor did we expect to see *mathematical* arguments advanced in order to support *epistemological* positions. Hence, the point of view from which we posed our research questions had to change. As our students changed, so did the way we saw them. In other words, we also needed to fold back, to see their words and actions in a wider context. There, however, the changes that we sought to understand became quite problematic to describe.

MARCI

In her journal entries for October 1995, Marci tackles, for herself, the challenge to describe a major change, just as it took place. It was Marci, recall, who personally built the group's main images, the torn paper arrays that anchored the central argument; and it was Marci's own "Oh, no! Don't say that!" that marked the recognition that the group's first number system failed.

> 10/12/95
>
> Wednesday's class was great.[46] We had to make up our own counting system using the symbols O, A, B, C and D. We used the blocks, piggies, longs and flats, but it was with 5 as the highest number. What a mind-boggling experience. It took our group a while to understand the actual problem and then it took a while longer to get started. But once we did, things started to go well. We had a whole system worked out and then realized that it didn't quite make sense. So we altered it just a bit. Our new method worked out fine and we got to the point that we could add and multiply. I wanted to try division, but others were not ready to go that far by the time I had to leave.

The second number system passed its main experimental test after what appears to be a minor change—but is that *all* that Marci learned from Katie and Laurel's argument? Here Marci writes again, exactly two weeks later:

> 10/26/95
>
> We are back to our normal number system. Working with the different number system was very challenging and extremely fun. There were a few frustrations, but not too many. Our group moved right along, doing subtraction, division, and eventually, working with fractions. At first I thought working with fractions would be quite difficult. But with our system, we found that fractions were not really hard and we got so we could add and multiply and compare and reduce. My overall thoughts on the new number system are that there is a certain reality to mathematical procedures that occurs everywhere. It's like a law of physics, it is true whether it be on Mars in 1695 or on Saturn in 2019. The point, I believe, of this project was to dig deep into the cor[e] reality of arithmetic and why certain things are done. Algorithms were developed and answers obtained as we waded through the sea of unknown numbers—or were they letters??

How, however, do we navigate this unknown sea? Now Marci tells us, for the first time in her journal, what is new for her, exactly what is new about it, and precisely where it came from:

> An assessment was given on October 23.[47] This was a good assessment. I'm not one to visualize or ask many questions why, but this time I really focused on why the problem worked and actually came up with a visual representation. I was rather impressed. It looks as though the rest of the group is helping me to see beyond regular class algorithms and focus on why, how, and what does it look like?

This quiet text, the first time we saw it, brought the drama into focus for us. Indeed, we learned from Marci that it is not enough to construct representations, make connections, and solve problems. You also need to build a theory, and for Marci, building theories starts with images, and then, through these images, proceeds to construct texts that can support conclusions strong enough to stand as universal principles.

DESCRIBING CHANGE

In the context of social anthropology, Clifford Geertz (1995) takes up the vexing problem of describing change. Like us, he does not find this question simple:

> Suppose, having entangled yourself every now and then over four decades or so in the goings-on in two provincial towns, one a Southeast Asian bend in the road, one a North African outpost and passage point, you wished to say something about how

> those goings-on had changed. You could contrast them then and now, before and after, describe what life used to be like, what it has since become. You could write narrative, a story of how one thing led to another, and those to a third: "and then...and then." You could invent indexes and describe trends: more individualism, less religiosity, rising welfare, declining morale. You could produce a memoir, look back at the past through the blaze of the present, struggling to re-experience. You could outline stages—Traditional, Modern, Postmodern; Feudalism, Colonialism, Independence—and postulate a goal for it all: the withered state, the iron cage. You could describe the transformation of institutions, structures in motion: the family, the market, the civil service, the school. You could even build a model, conceive a process, propose a theory. You could draw graphs. (p. 1)

Against what fixed backdrop, in other words, might motion be discerned? Geertz (1995), facing the same dilemma we confront, lowers the boom:

> The problem is that more has changed, and more disjointly, than one first imagines. The two towns of course have altered, in many ways superficially, in a few profoundly. But so, likewise, has the anthropologist. So has the discipline within which the anthropologist works, the intellectual setting within which that discipline exists, and the moral basis on which it rests....When everything changes, from the small and immediate to the vast and the abstract—the object of study, the world immediately around it, the student, the world immediately around him [or her] and the wider world around them [both]—there seems to be no place to stand so as to locate just what has altered and how. (pp. 1–2)

Admitting even this—and here he makes a central point—he believes that he can still proceed:

> To form my accounts of change, in my towns, my profession, my world, and myself, calls thus not for plotted narrative, measurement, reminiscence, or structural progression, and certainly not for graphs; though these have their uses (as do models and theorizings) in setting frames and defining issues. *It calls for showing how particular events and unique occasions, an encounter here, a development there, can be woven together with a variety of facts and a battery of interpretations to produce a sense of how things go, have been going, are likely to go.* Myth, it has been said, I think by Northrop Frye, describes not what happened but what happens. Science, social science anyway, is much the same, save that its descriptions make claim to solider grounding and sounder thought, and aspire, sometimes, to a certain dispassion. (p. 3)

For us, the narrative that tells not simply what has happened but also how it happens, needs to grow, as we have learned from Marci and the others who, together, brought about the changes she describes, from the demand for rigor that brought Marci back, weeks later, to deep questions about sense, validity, and meaning.

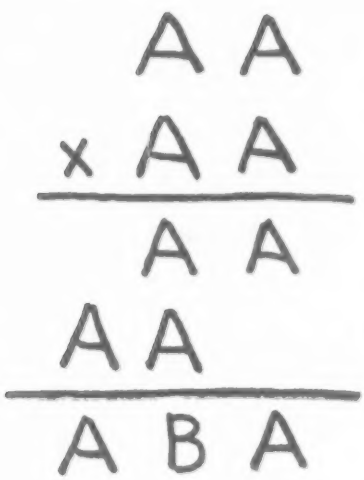

FIGURE 5.2. Marci's written representation of AA × AA.

WHAT CAN HAPPEN? HOW?

We focus here on the demand for rigor that has helped to shape our group's performance, and on how, concretely, this demand plays out in practice, both within our students' drama and beyond it. In a key exchange, just two minutes beyond the drama we have analyzed, Holly shows us clearly, for the first time, *why* this group has pressed so hard to build convincing reasons for their choices. At this particular moment, the members of the group, in different ways, will make their first attempts to build procedures for arithmetic, using the number system they have just defined:

Marci: Did you multiply?
Katie: No, I was doing subtraction. You've got BB minus AO.
Marci: I'm going to try multiplication. *[Writes down AA times AA.]*
Mandy: Scary.
Holly: Oh, you're going for a hard-core multiplication.
Katie: A times B, first, is B.
Marci: A times A is A. A times A is A *[follows the multiplication algorithm]*. Now you just add, A, B, A.
Holly: How in the heck are you going look at that and say…Oh yeah, I forgot…well…*[She is referring to the written multiplication algorithm, talking about shifting the second line of multiplication one place value to the left.]*
Mandy: Because if you're used to just using *your* number system. But if they've only ever had these five symbols…
Holly: But they're never going to look at that and say, oh, yeah *[moves her hand to the left over the written work to represent the shift in place value]*…Got to put this in…Leave a space there.

But, we outsiders have to ask, exactly who are *they*? The next lines suggest that every member of the group knows very well.

Marci: Yeah…Why not?
Holly: I wouldn't be that smart.
Laurel: Not yet. I can't even figure out how to add yet.
Mandy: If you were taught that way. They can't add and they can't do anything until they understand the numbers.

Now we know, too: *they* are the members of the audience these players now rehearse for—the children they will teach. Hence, it is *their* need for convincing explanation, as demonstrated by the Gang of Four, that drives these future teachers' inquiry. Lest symbols still mislead them, Laurel now asks to slow the pace:

Marci: So BA times CD. D times A is D. Ooh, what's B times D?
Laurel: Everyone's moving faster than I am.
Holly: Same with me.
Laurel: You guys, I know that this is an exciting discovery, but my mind just doesn't move that fast. I just need to go slower. All of the sudden you're multiplying, and I'm going, Wait a minute…
Marci: B times D is…
Holly: Yeah, how are you going to do this? *[Referring to Marci's multiplication.]*
Marci: Okay, B times D would be CB.
Holly: No, no![48]
Katie: Well, it's just memorization.

Here Katie, who has never favored memorized routines as substitutes for explanations, excuses Marci's imprecision. Next Holly, who first questioned the symbolic play, drives home the point *she* sees as central: that counting, alone, can *never* tell you how to multiply.

Holly: You can't multiply with this table at all *[referring to counting table]*.

Within two further minutes, led by Katie, the group locates the idea of multiplying in constructions with their blocks. In other words, they fold back[49] to concrete images, both to motivate and to explain the algorithms they will build. The concrete images, in other words, just like the pictures that the Gang of Four had drawn to represent their towers, are the things we need to organize in order to construct convincing explanations.

In Holly's words, counting and arithmetic reflect what she calls "gut realities," and the changes in her thinking, as she has described them, grew from recognizing and then working with those gut realities. As she shows here, through her own actions, the gut realities emerge through work with

concrete images, images of blocks, which embody adding as combining and multiplying as repeated adding. These concrete images, in other words, control her actions with the symbols.

Marci: B plus a D is an AA. So you go A *[writes A below right column]*, A [writes *A above left column]*. So B plus a D is AA...so yeah. That works. A B plus a D is an AA, again *[points to top A]* is a B. Which is a B, A. *[Writes B then A, right to left.]*
Katie: So you can carry.
Marci: So, yeah, you can carry. Exactly, that's just what I did. Did you get that?
Holly: Yeah. But I have to go that slow to understand it. Because you first start going off, I'm like whew...
Katie: When we were little we started off with...
Laurel: I have to do that too. *[Points to Holly's drawings.]*
Holly: We're sitting here drawing pictures.
Mandy: It's funny for me because I'm really...and I can do this...
Holly: Oh, see, I can't. I have to see it.

As we build them, abstractions need to be held tentatively while we check them against concrete images. This, as Holly's words and actions emphasize, is how the "gut realities" play out in practice: Without concordance between images and symbols, we have no way to understand symbolic processes, and this is what she says she has to *see*. The checking here must hence include the process, not just the result, based on the recognition that the process and its products constitute each other. This final observation, indeed, is what Laurel and Katie fought for with such focused energy when they folded back to longs to raise a key objection. That other members now concur, through their continued practice, reflects a fundamental change.

For us, our students' drama seems to constitute a single, complete action, leading to a fundamental recognition. For our students, however, it is also a rehearsal for performances to take place in the future, in classrooms that they will help to shape, where *children*, especially, will be actors, in turn rehearsing for their later lives. Against this backdrop, Holly's reflection that her understanding is a work in progress seems especially significant. The "gut reality" of mathematics in the making, as teachers and children first write, then reconsider, and then rewrite it, remains subtle and mysterious, despite the progress we have made. As Holly suggests—and we concur—we may have only just begun to fathom it. In this connection, read against her actions, Katie's own comments on the number system task will offer some important clues about the way these students see their mathematics, and themselves, as subjects in the making.

KATIE

In mid-December, almost two months after she and Laurel made their winning case, Katie submitted her first-semester portfolio.

> Task
>
> Making up our own number system
>
> When we were asked to make up a new number system without any numbers or symbols that we knew already I was lost. I had no idea how to represent a certain amount with anything other than our own number system. We were told that a long was equal to "the number of digits we have on one hand." (We were not allowed to mention any of our normal numbers.) We were told that we could use the symbols O, A, B, C, and D. It took us a long time and a lot of discussing but we finally came up with a system that we liked and felt worked. We started out stating the placement of the symbols. The column farthest to the right was the piggies column. Next to it on the left was the column for the longs, then came the flats and then the solids. Our counting went something like this:

A	AA	BA	CA	DA	AOA	AAA	ABA	ACA	ADA	BOA
B	AB	BB	CB	DB	AOB	AAB	ABB	ACB	ADB	BOB
C	AC	BC	CC	DC	AOC	AAC	ABC	ACC	ADC	BOC
D	AD	BD	CD	DD	AOD	AAD	ABD	ACD	ADD	BOD
AO	BO	CO	DO	AOO	AAO	ABO	ACO	ADO	BOO	BAO

> starting with the left column and reading down it and then moving on to the next column.
>
> Our number system also needs to be consistent while doing addition, subtraction, multiplication and division. My group was lucky because our system worked the first time.

This careful exposition emphasizes the final result, which is seen as true because it holds, and how the group, because it held, gathered around it. Only the sentence "It took us a long time and a lot of discussing but we finally came up with a system that we liked and felt worked" refers to what the tape clearly reveals: that Katie and Laurel, jointly, had to struggle for the floor to tell a counterstory; to demolish, by direct experiment, a system that had been developed, in detail, by all the other members of the group. Then, holding the floor with careful tact while thinking on her feet, Katie, first summarizing Laurel's doubts and then molding them into a closely evidenced critique, finally enabled Holly to provide, *based on the system that had failed and on the discourse that surrounded it*, a clear solution that the others could accept.

Similarly, the inservice teachers quoted in Schifter and Fosnot (1993) emphasized agreement and empowerment rather than what might well have been sharp battles to achieve those ends. Here we cite one such reflection, by a teacher named Julie, for comparison.

> I have at various times in the past seriously studied Piaget...yet I never really knew what they meant by "construct knowledge from within." I'm getting it now because [I'm aware that I] have...constructed knowledge for myself. (p. 62)

In contrast, Katie declines to step back from her experience to comment on its implications but instead presents her findings with assurance and precision. Rather than *mentioning* empowerment, in other words, Katie *enacts* the power she has won.

To understand exactly *what* was built—an agenda that Katie has shared and furthered deeply through her actions—we have postulated frames and then constructed texts around them, through which our students' words and actions, detail by detail, have become a narrative. These frames and texts now anchor our abstractions, just as the blocks provided concrete anchors for the students as they built personal abstractions.

THE PROBLEM HAS A STORY OF ITS OWN

If what we can remember builds from symbolic representations, coded as frames and texts, then the frames and texts we carry with us constitute what we have *been*. The way we use these frames and texts, by contrast, helps to show us who we *are*. This showing, though, takes place, as we have seen, within a telling, a performance, built from images and texts, which themselves become subjects for scrutiny, for argument, and for revision. In this sense, past knowledge is rebuilt, based on present experience. For our students, the current experience, in the session we have studied, strikes them, in Katie's words, as a fundamental *problem* that they need to overcome.

The students' drama, indeed, narrates, through images and texts, the story of a problem. Taking that problem as a subject,[50] we can talk about its role, its needs, and its effects on different people. Through the story of the problem, we can also see most clearly some important changes in the ways that our five actors work and think.

The problem here began, as Katie and Laurel frames it, as a fundamental difficulty with the representation of a long. Their framing of the group's *mathematical* dilemma would set them, as the tellers of a counterstory, at the margin of their group. Viewed from this perspective, the breach that Laurel and Katie made challenged not only the group's *cognitive* structures (frames and frame instantiations) but also its *social* structure. Indeed, Laurel's nonverbal actions, such as rising from her seat and moving toward the center of the circle, embody physically her movement from the margin as she presses an assertion that three other members of the group (Holly, Mandy, and even Marci) have, so far, excluded from consideration.

Laurel and Katie's breach, in other words, seems to produce what Turner (1974, 1988) calls an "anti-structure," or "communitas," as much a process as a state in

which all members of the group will have to share and share alike. This quality of interaction, in which a dominant discourse is broken down by forceful action from the margins—in which the group, after the breach, must function as a team of equals—is called "liminal,"[51] after the Latin word for "threshold."

The threshold metaphor seems especially apt here, because it helps us focus on the group's response to its dilemma, in particular the role, in that response, of what Katie has called the "problem." Indeed, in Katie's view, it is that problem, no more and no less, that first triggers the conflict; next leads to a crisis; and then, pushes the whole group, now as equals, into unknown water.

At that moment, with several key frames under fierce critique and with the group in turmoil, Katie triggered what would soon become a crisis in the *problem's* life, not just the group's. "We have a problem," she declared, and then she named it: "We don't have a nothing." Katie's naming of the problem, in the terminology of Epston and White (1990) *externalized* that problem, set it at some distance from the group's perceived identity, in the sense that the members of the group could now, if they so chose, begin to function independently of the mathematical and cognitive choices that had led to their dilemma. From this point on, the problem's story would be one thing, and the story of the group, as it reconstituted both its cognitive and social structures, would be something else.

The problem that Katie named, indeed, threatened the social structure of the group. Marci continued to assert that D piggies adequately represented a long, until Laurel extended Katie's negative critique from an examination of counting constructions to a larger analysis, through place value shifts, of how these must emerge from *meanings* of the number symbols. This joint performance symbolically reshaped the problem as a *focused question* about which quantities the symbols O and A should represent, and Holly (a spokesperson for what had been the dominant discourse) quickly found this question helpful. In a sense, the problem, too, had undergone a liminal experience, first by being isolated from the group and then by being made to change its status.

At this point, several group members, significantly, began to draw and write. Viewed as a text, the story of the crisis, too, had been rewritten. In particular, we no longer read Marci's first experiment ("Oh, no! Don't say that!") as a failure but, rather, as a meaningful success.

THE PORTFOLIO ESSAYS

Katie completed her December portfolio with an essay that described the dominant discourse of school mathematics, along with what she felt was wrong with it: no roots in experience, no reasoning, and hence, for her, no truth. And she named the challenge, which was to help children to understand, by building meaning. We all know that most school mathematics, and

much college mathematics, has been just as Katie claimed. The hard step, both for Katie and for us, was and is to describe a strong alternative, concretely, as it works in practice.

> December 20, 1995
>
> I soon discovered that there was a lot about mathematics that I took for granted. I never asked the question of why things worked the way they did, I just always believed that they did and put my trust in the teacher to not teach me anything wrong. During this past semester I have found myself constantly asking why a concept works the way it does. Why, when dividing fractions do you multiply the top fraction by the reciprocal of the bottom fraction to get the answer. I always knew it was how it was done but I never understood why. Simple mathematics that I thought I knew like the back of my hand became new and exciting to me. I think that if all children were taught in this manner from the beginning of his or her education, the love of mathematics would grow in schools. Students wouldn't be so confused. They would understand why.

Just as in the drama of October 11, it was Holly who would step from clear critique to concrete action. In her final reflection, she began to write about what teaching needs to *mean*, and then commited herself *in writing* to prepare herself to act on what she now had come to understand:

> We have approached math differently than I, and many others, ever have. I feel that by doing so I have learned so much more that will really apply to me as I teach elementary math concepts. However, I have recognized this last semester that I do not really teach, I have the difficult responsibility of helping them [the children] to recognize and teach themselves how math works....

Here we would like to emphasize the final sentence. Surely this sentence strikes directly to the core of this group's struggle, and of our own efforts to make sense of it. To make that sense, we need to take the measure of what building mathematics (from its base in folklore, children's rhymes, and concrete images, to its summit in the deep realities, whether in our guts, as Holly put it, or in universal principles, as Marci emphasized) entailed for these five students. Here is Marci's final look:

> As I look back on the semester and think about Math 305, a lot of thoughts cross my mind. This semester of math produced a greater understanding of the fundamentals and basic concepts of math. Most important, it helped me see math in a way I've never seen it before. As written in my journal, I have never been a visual math leaner. I learned math by formulas and equations and never worried about why or how. The beginning of the semester was very challenging for me. In fact, the first two assessments show my inability to really explain why I did what I did. It's amazing that in the course of one semester my thinking in math could change so drastically. And not only do I think of math differently, but it's interesting to find me

thinking of life in a much more illustrated or visual way.

The first time we used Unifix blocks, I didn't know what to think. When we were given a task, I immediately tried to think of what formula would help me figure out the answer. When we first worked with groups and had to find answers to problems, we all tried to think of what the problem had to do with our previous learning of math. Along the course of the semester, our group started to look at math in a different light. I knew that the problems we did had simple enough algorithms to figure out the answer, but it was more interesting to use the blocks or rods and really look into the heart of the problem.

Although I still struggle with wanting a plain and simple equation or formula to learn math, I have learned that there is more to math than that. Granted, I'm not a very visual (or should I say creative) thinker, in fact, I have never thrived on creativity, but I have learned to think more in this way. This class has helped me realize that people look at math in different ways. As a teacher, that is important for me to know. Had I not realized this, all of my students would have had to learn math in the same way I did, memorizing formulas and never asking why.

Here Marci took a comprehensive look at her own mathematics, from the summit of her journal, where she aligned her personal discoveries with laws of physics, to the base, of images and reasons, which we all need to build, beginning as small children. Indeed, in its own way, each student text confronts the stark reality of schools that will not, or do not even know how to, take their learners seriously. Our students staged this confrontation from the platform of a classroom that has clearly done so, in rehearsal for the work which they will later do as partners of the children in their classrooms.

CONCLUSIONS

We began this investigation, before our students had arrived, by postulating a set of cognitive structures, called "counting constructions," to provide images[52] for basic arithmetic constructs, such as sums and differences, products and quotients. The images we chose emerged in close relation to the physical materials, in this case blocks, that we had planned to use in order to explore procedural arithmetic. Expressed as frames, with terminals, defaults, and built-in constructive processes, counting constructions allowed us to obtain—and our analysis supports this claim from many angles—a clear, compact description of the mathematics that we and our students investigated, as it grew from concrete images toward algorithms clearly based upon these images. These mathematical explorations, as our students' arguments revealed, supported a larger argument, in which our students challenged the stock story that a place value system needs to have a symbol representing "nothing," by building a coherent system without using such a symbol. This initial system failed, *not* because it failed to make internal sense, but because, on two related counts, it failed to represent the *blocks*. This failure, and the group's

construction of a second system, which includes a symbol representing "nothing," a system that precisely represents the blocks, challenges a second stock story: that constructs "hold" because their underlying principles "are true." On the contrary, our students' actions, as we view them, urged that principles become true[53] when their consequences can be shown to hold.

In a similar way, our frame descriptions seem less tentative and less preliminary than they may have seemed to us at first, now that we have used them as foundations for a *thick description* (Geertz, 1973) of our students' complex, far-reaching, dramatic interaction.

Our work throughout, built from the view that people "reconstruct" their culture's mathematics as they learn it—much as children reconstruct their culture's language as they learn to speak it—because, as Chomsky (1975) argues forcefully in the domain of language, we seem to have been born with the capacity to do so, not in a uniquely determined way but according to designs and principles that allow us to construct a particular mathematics—or a particular language—triggered by personal experience, by social interactions, and by what seems to be a universal need to build descriptions of the world that will make sense and also hold, both for ourselves and others. Neither the personal construction nor the social interaction may be omitted from consideration if we wish to understand and to support our students' growth. So these students emphasize, through their sharp, thoughtful critique of what they have encountered as they tried to build their mathematics—and, as we have shown, their awareness of their own power, their personal capacities, and their special stakes, as actors and as authors—in the social world they help construct and reconstruct, through what they think, say, write, and do.

Appendix A

Mathematics 305, Fall 1995

We taught one section each. Each section consisted of about 20 students, organized into four roughly equal groups.

THE ROOM

The classroom, about 20 feet square, had movable chairs with wide arms for writing, one small table at the front, an overhead projector (with screen) at the front right corner, blackboards on all four walls, no windows, and no other furnishings.

HOW WE WORKED

On the first day of class, we asked our students to fill out index cards to let us know a little bit about them. We especially invited them to sketch their recent mathematics background and to tell us something helpful about what they hoped to gain from their participation in the course. Their responses often told us how they felt about the official mathematics that they had encountered and, more personally, about how confident they felt about their own mathematics.

Some, indeed, seemed very confident, while others emphasized concern about strong negative impressions about mathematics and about their own ability as mathematics learners, which had existed, several reported, at least since middle school. We assigned students to groups in the first week, seeking maximum diversity within each group. These groups remained together through the term.

Each class met twice a week, in 100-minute periods. Such class periods, although unusual, were, we believe, essential for extended exploration and reflection.[54]

MATERIALS

Physical materials would trigger most of our explorations. We worked with Unifix cubes in two colors (a total of about 2,000 cubes, in clear plastic bags of 200 each), Cuisenaire rods (the equivalent of two standard classroom sets, in plastic bags, one to a group), pattern blocks (four buckets), and base 10 blocks (two classroom sets, packed in their standard boxes). We needed every block we had, given the complex explorations that our students undertook. Each day we carried boxes of materials from our crowded offices to our classroom and then back again.

Because so many blocks fell to the floor, we sometimes invented tasks that required working on the floor. Our students dressed accordingly and often pushed their awkward, metal chairs aside to work more freely.

THE SYLLABUS

Our syllabus, distributed on the first day, described the special nature of the course, reviewed our backgrounds, and explained our particular approach to teaching, learning, and assessment (see Figures A1–A4.)

Faculty: Bob Speiser (10-11:50) and Chuck Walter (8-9:50)
Texts: National Council of Teachers of Mathematics, Curriculum Standards.
Herbert Ginsburg, Children's Arithmetic

This is the first part of the two-semester sequence 305–306, designed to help prepare education students to teach mathematics in elementary schools. Although we will explore basically the same mathematics that the regular sections will cover, our approach will be quite different. In particular, we will base our daily planning on what we learn about your work and thinking as the course proceeds, so we'll be working closely with you, with lots of personal attention, lab support, and very active interest. Hence we strongly recommend that you plan to take both semesters with us, to maintain continuity and extend the benefits as long as possible.

Who we are. Both of us began as research mathematicians, but with strong interests in education throughout our careers. We spend each Thursday at the Longview Elementary School in Murray, working with the whole school to strengthen its mathematics curriculum, in a collaboration now in its third year. We also collaborate with colleagues at the Center for Mathematics, Science, and Computer Education at Rutgers University, to help organize a national network of teachers, principals, educational researchers, psychologists and curriculum developers, to help implement some of the best ideas in mathematics education. Our research has focused on our own students as well as on school children. In particular, this course will be a major focus for us, not just as teachers, but as scholars, interested deeply in how you think and learn.

Goals. We're interested in making sense of children's mathematics, from several viewpoints: personally, in terms of our experience; socially, through the mathematical understanding, language and technique shared in society; through children's eyes, as they explore mathematics in their own classrooms. We'll weave

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FIGURE A1. Syllabus, p. 1. Mathematics 305, Fall Semester 1995.

our course work, like a braid, from three main strands:

- Discovery: experiencing personally how mathematics happens.
- Language and technique: how we communicate and understand, especially through shared language and procedures.
- Principles, connections: key ideas which help to organize and verify our thinking.

In this course, we will emphasize discovering key mathematical ideas through exploring and "unpacking" concrete, interesting problems. By "unpacking," we mean focusing discussion on how we communicate mathematical thinking, how the ideas we're finding fit together logically. One of the beauties of mathematics is the way it can be organized so well around a surprisingly compact core of "big ideas", ideas which help us organize our thinking, while they guide and shape curricula.

The course. We'll develop the mathematics, itself, in our classroom, through direct exploration, followed by careful unpacking. Assigned readings will provide additional perspectives on the mathematics, from viewpoints more closely centered on children and working schools. We'll also explore some classroom videotapes, unpacking children's real-life mathematics from viewpoints given by our own learning and discussion. Connecting with children's explorations is what makes our model new–we emphasize both personal discovery and sharp observation in order to prepare you best for the kind of listening and learning we do when we teach, and which we hope you'll do in turn, when you lead your own mathematics classrooms.

Because class discussion, not a textbook, is really going to be the starting point, you'll be doing a lot more pulling together in groups and on your own than in a lecture-and-discussion format. Tie your discoveries together in carefully prepared notes, work the assigned problems carefully, and think through the questions you will ask us, so we can respond as effectively as we can. Come to the labs we'll schedule, especially on Tuesdays, when we'll have an open house most of the day, with lots of materials and help available. Think about children's mathematics in relation to your own, and in relation to ours, which we'll be sharing as generously as possible. Dig deep into the homework explorations, and come talk with us – your learning, this year, is our research subject, and we're eager to learn a lot about your growth and thinking while we back your explorations.

FIGURE A2. Syllabus, p. 2. Mathematics 305, Fall Semester 1995.

PORTFOLIOS

Each of you will prepare and submit a portfolio which contains the record of your course work, and your thoughtful reflections on that work. Discuss carefully the development of your own mathematical understanding, and relate your understanding to what you've learned about how children think. Your course grade will be determined by the content and quality of your portfolio.

The record of course work to include in your portfolios consists of the following items.

- **Assessments** (7)
  There will be seven assessment tasks spaced at regular intervals throughout the course, which each of you will complete. Each task will center on significant mathematical issues, your own or children's, which we are currently exploring. Although relatively short (requiring 25 minutes in class to complete), each assessment will be carefully designed to provide critical insight into your own mathematical understanding, your ability to effectively articulate that understanding , and/or your understanding of children's mathematical thinking.
- **Exhibition** (1)
  Each of you will choose one assessment task and produce a detailed development of that task which highlights the mathematical and pedagogical issues central to that task. This work will be presented in a format which allows it to be exhibited and examined by the community in a poster session devoted to these exhibitions.
- **Final Examination** (1)
  You will complete a final examination, at the time published in the University Final Exam Schedule. This examination will be weighted as equivalent to two regular assessments and, in a way congruent with the regular assessments, it will center on your understanding of central mathematics as well as on your ability to understand children's mathematics in the classroom
- **Interviews** (2)
  You will be interviewed by the instructor on two occasions. These 20-minute interviews will be videotaped and the tapes will be given to you to provide a

FIGURE A3. Syllabus, p. 3. Mathematics 305, Fall Semester 1995.

concrete basis for your reflection on progress and/or problems. These taped interviews, as well as your comments on them, are to be included as part of your portfolio.

- **Journal** (1)
  You will be asked to keep a personal journal in which entries, at least weekly, provide a record of your own reflective process as the course progresses. Reflections on your work, the development of new insights, and especially, thoughts on children's mathematics are among the appropriate topics to describe.
- **Favorite Problems**
  You should present a short list (3 or 4) of your favorite problems from from the homework you have done, along with a description of the attractive features of each.

The reflective part of your portfolio should include the following items.

- **Reworked Assessments** (2)
  You may rework any two assessments, if you wish, and submit these along with comments on the new work versus the old. These comments should describe carefully what has been learned in the process of critically examining your work on these assessments.
- **Selected Journal Entries**
  Although the entire journal is to be submitted as part of the portfolio, you should compile a few pertinent entries which give the flavor of the journal and focus your reflections on your work and growth.
- **Essay** (1)
  A brief essay of at most one page, summing up your experiences in the course. This essay should tie together the various threads that are presented in the rest of the portfolio.

FIGURE A4. Syllabus, p. 4. Mathematics 305, Fall Semester 1995.

FIRST TASK: BUILDING RECTANGLES WITH SQUARE TILES

This task opened the *Teaching Standards* (NCTM, 1991). We assigned it on September 6, the first class meeting, via a handwritten overhead transparency.

> How many rectangles can you make with 12 square tiles?
> How many can you make with 16?

We distinguished, for example, between a two-by-three rectangle and a three-by-two rectangle, and, to avoid trivializing the next question, we restricted attention to rectangles with more than one tile.

> What numbers of tiles will give the fewest rectangles?

We used single Unifix cubes for tiles[55] and invited students to construct rectangles physically.

SECOND TASK: COUNTING TOWERS

This task is the problem tackled by the Gang of Four. We assigned it on September 11.

> How many towers, three blocks high, can be built from Unifix cubes with two colors to select from? Can you convince me that there are no more and no fewer?

After our students worked on this task for one hour, we showed the tape of the Gang of Four's performance, more than once, and invited our students to discuss it.

On September 13, our students carefully investigated towers four blocks high.

INTRODUCING CUISENAIRE RODS

On September 13, we distributed the equivalent of two full classroom sets of Cuisenaire rods and invited students to build freely with them. After more than half an hour, we invited them explore the ways in which the different rods might be related. We also agreed on standard color names, in order to refer to them.

IMPROVISATION: ARITHMETIC WITH RODS

With the white rod, for the moment, representing 1, we posed questions such as:

> What does the blue rod represent?
> What does the black rod represent?
>
> Add 4 and 3, using the rods.
> Add 4 and 7.
> How do you know that you are right?

Next, we posed questions that were *independent* of the value of the white rod.

> Is a purple plus a dark green the same as a dark green plus a purple?
> How do you know?
> Does this work for any two colors?

Next we considered multiplication, with the white rod representing 1.

> Find 3 times 4 and 7 times 4, using rods.

Some students gave the product as a line of rods, which they could measure; others found it as a rectangular area, which they could structure in various ways. Noting this, we took larger factors, and recycled.

> How might rods best represent a product?

We then noted that almost everyone, spontaneously, had built a staircase, and that most had also built a double staircase.

> Suppose we have a staircase which starts with a white rod and ends with an orange rod. How many white rods are needed to cover the staircase?
>
> How would you find the sum $1 + 2 + \ldots + 9 + 10$?

Variations followed, such as finding the number of games in a round-robin tournament.

FIRST ASSESSMENT

On September 20, we gave the first assessment (see Figures A5 and A6) and unpacked it.

Math 305 Professors Speiser and Walter September 20, 1995

Assessment 1 (20 minutes)

Use the grid on the other side of this sheet to find the product 13 x 28. Choose your rectangular representation carefully to take advantage of the heavy lines in the grid as well as the lighter lines, so as to shorten your work. Explain carefully, as in a letter to a friend, why you made the choices you did, and how those choices help to make the counting process for the product more efficient.

FIGURE A5. Assessment 1, p. 1. Mathematics 305, Fall Semester 1995.

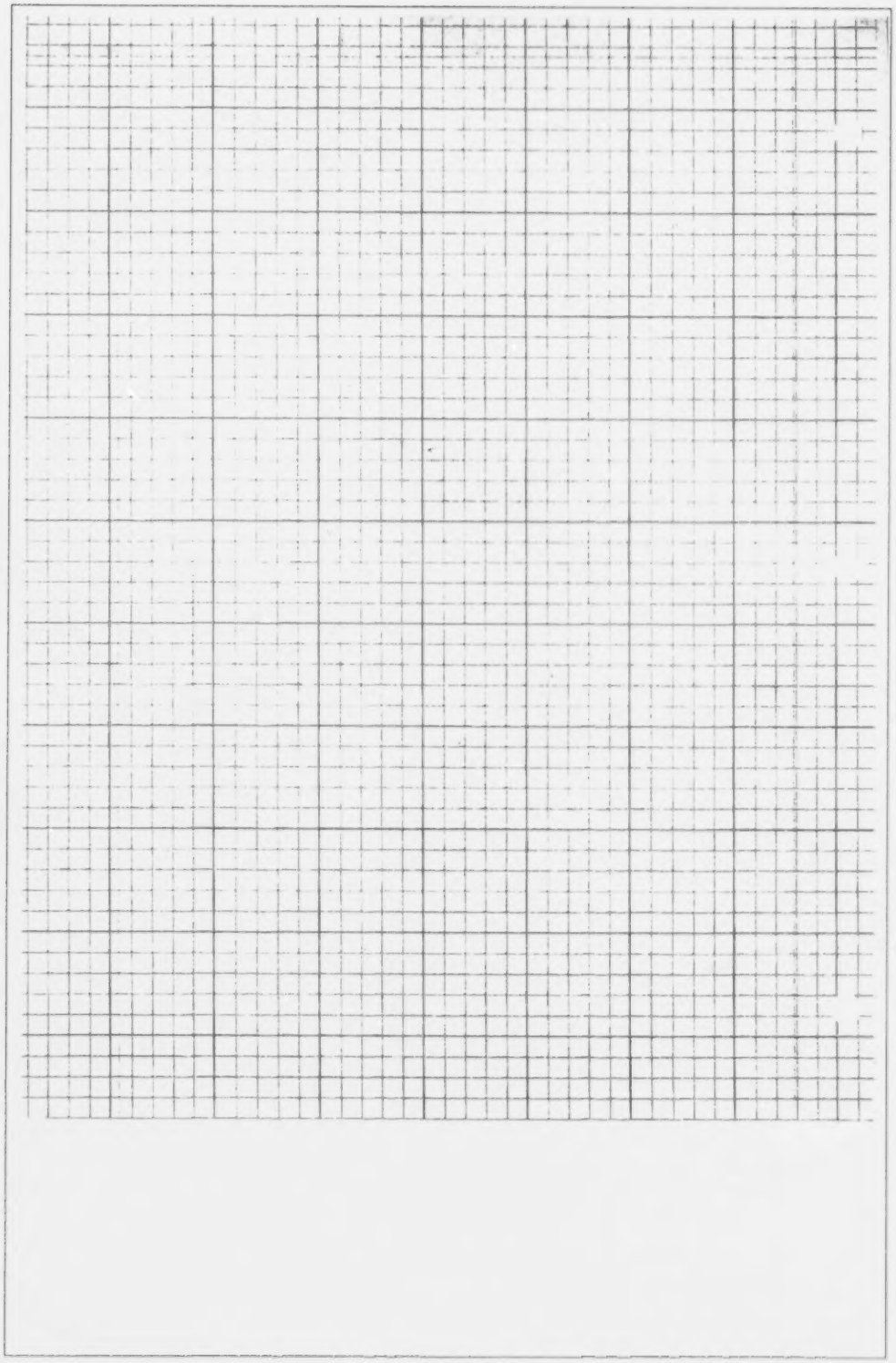

FIGURE A6. Assessment 1, p. 2. Mathematics 305, Fall Semester 1995.

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THIRD TASK: REPRESENTING NUMBERS

Then we gave a task.

> If the red rod represents 3, what does the purple represent?
> The black?
> How do you know?

We followed these questions with further variations.

FOURTH TASK: SUBTRACTION AND DIVISION

> With the white rod representing 1:
> subtract 6 from 15.
> Then subtract 5 from 14.
> Why are these results the same?
>
> Using pencil and paper, subtract 19 from 101.
> Then take 17 from 99.
> Why are these results the same?

The next tasks invited us to compare repeated subtraction to finding fractional parts. [Note: This time, we specified no given value for any rod.]

> How many fours go into 16? (Use rods.)
> What is 1/4 of 16? (Use rods.)
>
> How many threes go into 16? (Use rods.)
> What's 1/3 of 16? (Use rods.)
>
> Why are the last two results similar?

FIFTH TASK: EXPLORING BLOCKS

At the end of class on September 25, we offered base 10 blocks to our students and invited them to think about what these blocks might be useful for. Very few had seen them.

On September 27th, we named the blocks. Both instructors, after class work in which students usefully assigned different values to given Cuisenaire rods, hesitated to suggest calling the smallest base 10 block a "unit." Quickly Speiser's students suggested "piggy," and it stuck.

How many piggies make up a solid?
How long a line of piggies would that give?
Can you build it in this room?

How do two flats compare with 173 piggies?
How about 3 solids vs. 28 flats, vs. 263 longs, vs. 2552 piggies?

Represent 375 and 2637 in three different ways with blocks.

How big a number can you represent with the blocks available to your group? Compare with Cuisenaire rods.

SIXTH TASK: ANOTHER LOOK AT MULTIPLICATION

On October 2, we let the piggy represent 1.

Find 34 times 28 using blocks. How might you represent the result, using the fewest blocks?

We made a rule to favor the subjunctive mood in posing invitations. By now, the need for careful explanation had become implicit in the classroom culture.

Find 34 times 38.
Find 34 times 128.

Build a *theory* which predicts how the last two products will work.

Test this theory on 77 times 235, by making your prediction first, and then the blocks construction.

How might your representation of the last product using blocks relate to a pencil and paper procedure? More precisely, how might you organize the count?

SEVENTH TASK: DIVISION

On October 2, we explored division using blocks. Sample tasks:

Using blocks, divide 52 by 17.
Divide 52 by 3.
What might the remainders mean, in each case?

Divide 2196 by 108.
What might the remainder mean?

SECOND ASSESSMENT

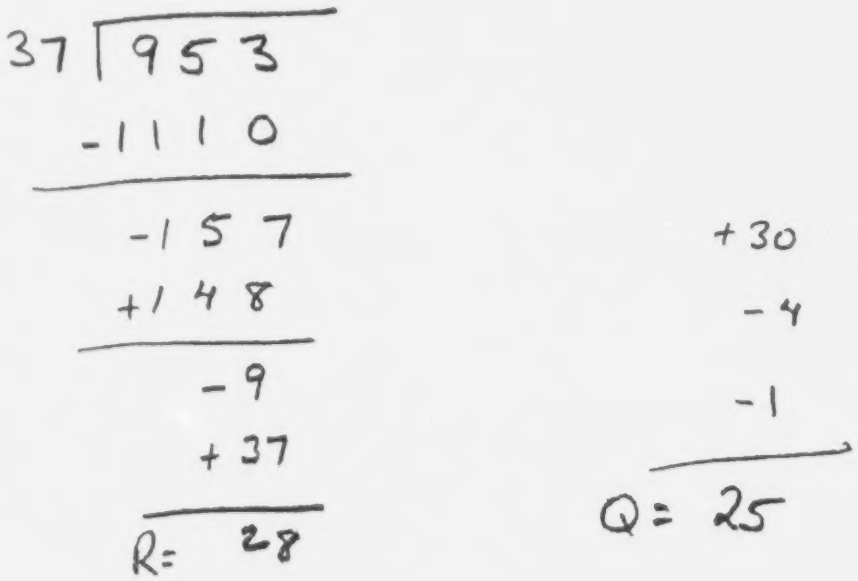
On October 4, we gave the next assessment (Figure A7).

Inviting our students to defend a child led to remarkable discussions. Several students hastened to recall past encounters in which teachers had rejected their ideas. Although no gender was mentioned in the assessment, our students often gave the child a gender as they wrote. Sometimes the child was female, sometimes male. The strength of several papers, whose authors argued passionately in favor of this unknown child, began to haunt us.

Math 305 Professors Speiser and Walter Ocdtober 4, 1995

Assessment 2 (30 minutes)

Here is a child's work on a division problem. Is she right? A colleague thinks that this child fails to understand the concept of division. Based on the evidence at hand, what would you tell him?



Please give your respose on the other side of this page. Use pictures if appropriate to help support your words.

FIGURE A7. Assessment 2. Mathematics 305, Fall Semester 1995.

IMPROVISATION: SIGNED NUMBERS

After unpacking the assessment, we proposed some further tasks:

> Find 55–34, and 34–55, with blocks.
> What is similar and what is different about these questions?
> What does subtraction mean in each case?
> How do you do it?

Then we introduced Postman Stories (Davis, 1980, Chapters 12, 14, 16) together with appropriate notation and terminology (Davis, 1980, Chapters 8, 9) for positive and negative numbers, which we reviewed informally—and lightly—as we went along. To conclude, we revisited the subtractions above, this time in the language of the Postman Stories. The latter continued through October 9, ending with discussion of a recent video of Davis teaching Postman Stories.

EIGHTH TASK: BUILDING A NUMBER SYSTEM

This task began October 11 and continued through October 23. On October 16, each group concluded, at the overhead, with a progress report. After these reports, work continued on arithmetic. For the reports, we posed several questions:

> What strengths and weaknesses do you see in your system?
> Advantages and disadvangtages?
> What are your specific goals, and how do you know that you are making progress toward them?

On October 18, focused on arithmetic, including fractions, we asked the following question:

> If a number is prime in your system, is it prime in mine?

The session on October 23 began with group project reports, again at the overhead.

THIRD ASSESSMENT

The last assessment's unknown child now received a name: *Zoe*. She wrote to pose a challenge to our students in response to their work on Assessment 2 (Figure A8).

Zoe is Greek for "life."[56] This surprising child, a fiction only in a very narrow sense, became a metaphor, first for the research team, then for the students. Zoe stood for our capacity to have surprising insights, wonderful ideas, and then to share them and to enact them, most especially in settings where ideas and insights might be valued and preserved.[57] The point of any "Zoe assessment" is to recognize strong thinking by a child, and then defend it.

Math 305 Professors Speiser and Walter October 23, 1995

Assessment 3 (20 minutes)

Dear Class,

after Bob, Chuck, and Tara showed me your last assessments, which were based on my work, have I got a question for you!! Look at this —

1000 - 33 = 967
999 - 32 = 967
998 - 31 = 967
997 - 30 = 967
996 - 29 = 967
995 - 28 = 967
994 - 27 = 967
.
.
.
970 - 3 = 967
969 - 2 = 967
968 - 1 = 967

WHAT'S GOING ON HERE and WHY???

Best,

Zoe

PS: I don't like borrowing any more than you do. take a look!

1000		999
- 33		-32
yeech!	967	---- easy

How about figuring out 723 without borrowing?
-168

Think about this for Wednesday.

FIGURE A8. Assessment 3.—Mathematics 305, Fall Semester 1995.

Appendix B

Mathematics 305, Fall 1997

In this appendix, we give versions of our tasks and assessments for the entire first semester of our preservice mathematics course (Mathematics 305). The second semester, which concentrates on probability, geometry, and computation, has a somewhat different flavor. Our course design assumes a class of 24 students, organized into four groups of six. New tasks demanded more materials.

MATERIALS

1. About 1,000 1-in-by-1-in builder's tiles (as for a bathroom floor).

2. About 2,800 Unifix cubes in two colors, snapped together into groups of 10 of the same color. Each set of 20 such groups is placed in a large Ziploc bag; each bag therefore contains exactly 200 cubes of one color.

3. Four "classroom sets" of Cuisenaire rods, packed in four plastic storage boxes.

4. Three "classroom sets" of base 10 blocks, with the longs and piggies packed in Ziploc bags.

We often work with these materials in unusual ways. We are especially interested in first understanding these materials in and for themselves (as in the Blocks

Frame of Chapter 2), and only then inviting our students to build important mathematics based on personal ideas and images that these materials suggest.

WORK IN PROGRESS

These tasks are here to sketch a work in progress, not to provide a detailed picture. Our course is still evolving; so is how we understand it. In particular, it seems unlikely that this book, or this appendix, could provide enough information about our course or our approach to enable someone to implement even a partial replication. Indeed, the mathematics, as we see it, is not in the tasks or the assessments any more than in the tiles, cubes, rods, or blocks we often use, but, rather, in people's minds, often, perhaps, in other forms than might be suggested here. We expect to prepare a more complete description soon.

To: **Math 305 Students**
From: Professors Speiser and Walter
Subject: First task
Date: September 8, 1997

This task is about building rectangles with square tiles.[1] To keep things interesting, we'll always work with at least two tiles. We'll consider these rectangles, for example, to be different, and we'll denote them as shown.

2 x 3 3 x 2

1. How many different rectangles can you make using exactly twelve tiles? Please explain why the number you have found is right.
2. Now try sixteen tiles. How many rectangles can you make using all sixteen? Please explain.
3. What numbers of tiles will make the fewest rectangles? Why is that?
4. Can you find a number of tiles that will allow you to make more rectangles than you can make with twelve tiles? Again, please explain.

Questions for further investigation:

5. What numbers of tiles will make an odd number of rectangles? Why?
6. What numbers of tiles can produce exactly three rectangles? Or four? Is there a pattern here?

[1] We'll use Unifix cubes.

FIGURE B1. Task 1. Mathematics 305, Fall Semester 1997.

To: Math 305 Students
From: Professors Speiser and Walter
Subject: Counting towers three blocks high
Date: September 10, 1997

Here we build towers using Unifix cubes in two colors, which we can think of as dark and light. A tower has a top and a bottom, and two towers are the same if they have the same colors, in order, from bottom to top. (We allow towers to be all dark or all light, as well as mixtures of the two.)

1. How many different towers exactly three blocks high can be built from Unifix cubes when there are two colors available for use?

2. How do you know that you have them all?

3. Can you convince me that you have all the possibilities, that there are no more and no fewer?

These questions are due to Robert B. Davis, Alice Alston and Carolyn Maher, researchers in mathematics education at Rutgers, who designed them carefully for children.

FIGURE B2. Task 2. Mathematics 305, Fall Semester 1997.

To: Math 305 students
From: Professors Speiser and Walter
Subject: First Cuisenaire rod explorations
Date: September 17, 1997 [Revised, September 18]

1. Take a few minutes to explore the rods informally. What might you notice about them as you experiment and build?

In what ways do the rods differ? Are all rods of the same color also of the same length?

How many different colors are there?

We'll need to name the different colors.

Please be sure to build some shapes which you find interesting.

2. Many children use Cuisenaire rods to understand whole-number arithmetic.

How might we add 9 and 6 using rods? How about 6 and 9?

How might you find 56 + 39 using rods? How about 39 + 56? Using the ideas you used to find these sums, can you explain why two such sums, adding the same two numbers but in different orders, must be equal? (Hint: Look back at your example. What do the rods which you are using count? How might the numbers to be added organize the counting?)

Explain how it could make sense to add rods, rather than numbers. For example, how might we add a purple to a dark green? Is a purple plus a dark green equal to a dark green plus a purple? How might you convince us?

How might we find 7 x 4 by using rods? What about 4 x 7? Can you explain why these two products (or any two products where we take the same factors in different order) have to be the same? (Hint: Look carefully at your example. What do the rods which you are using count? How might the numbers to be added organize the counting?)

Use rods to find 54 x 37, as efficiently as possible,and then justify your result. What features of the rods seem helpful here? Which don't?

If we organize the rods (one rod of each color) in order of increasing length, we obtain a staircase, beginning with a white rod and ending with an orange rod. How many white rods does it take to cover this staircase? How do you know?

Now find 1+ 2+ ... + 100. How might you explain what you have found?

FIGURE B3. Task 3, p. 1. Mathematics 305, Fall Semester 1997.

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3. The handshake problem.

We have a party with 15 people. Each pair of guests shakes hands exactly once—this is the social convention. How many handshakes will there be? How might you convince us?

4. How many rectangles are there in this picture?

How might you explain your result?

5. Cuisenaire rods were invented originally to help with fractions. Here are some first examples.

We claim that the light green rod is 1/2 as long as the dark green rod. Is this true or not? Please try to convince us.

Someone told us that the red rod is 1/2 as long as the yellow rod. What do you think? How could you convince us?

Someone else said that the light green rod is 1/3 as long as the blue rod. What do you think? How might you convince us?

Can you find a rod that is 1/2 as long as the blue rod? How could you convince us?

What would a rod look like in order to be 1/2 as long as the blue rod?[1]

[1] These final explorations are based on a workshop handout by Bob Davis, Carolyn Maher and Alice Alston, Longview School, Murray, Utah, June 1995.

FIGURE B4. Task 3. p. 2. Mathematics 305, Fall Semester 1997.

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To: Math 305 students
From: Professors Speiser and Walter
Subject: Base 10 blocks, I
Date: September 17, 1997

1. Explore the blocks informally: How do small blocks fit together to make larger blocks? How many are there?

2. What might we call the blocks? Folklore gives some names: the large cubical block is often called a solid; the next-smaller block, shaped like a square, is often called a flat; and the narrow, rod-shaped block is often called a long. Most of us who work with blocks resist, however, calling the small cube—which looks a little like a white Cuisenaire rod—a "unit", which leaves us with at least one decision. A few years ago, several 305 students suggested calling the small cube a piggy, as in the children's rhyme for counting toes. This sounded good—but perhaps you have a name which might seem better. There seems to be no standard usage here.

3. Blocks can help us represent numbers. How might you use the blocks to represent these?

11, 35, 105, 1998

(Hint: think about these numbers from different angles. Possibly you might want to represent them in different ways.) Further, what might be the largest number you can represent?

4. How might you use blocks to find the difference 3119 – 489? How about 1007 – 9? And 1000 – 2? Let's think a moment about explanations here. How do you see subtraction in these problems, and how do you use the blocks to represent this?

5. How might you multiply 54 x 37 using blocks? Perhaps you might want to consider different possibilities. How do you see multiplication here, and how do you use the blocks to represent this?

6. Let's reflect for just a moment on the last two questions. How does the way you work with blocks compare with how you might do the same arithmetic with pencil and paper? Perhaps your work with blocks can lead to something new.

7. To sharpen the last question, why not try dividing 543 by 11, using blocks. Is it possible to see division here in different ways?

FIGURE B5. Task 4. Mathematics 305, Fall Semester 1997.

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Math 305 Professors Speiser and Walter September 24, 1997

Assessment 1 (20 minutes)

Use the grid on the other side of this sheet to find the product 13 x 28. Choose your rectangular representation carefully to take advantage of the heavy lines in the grid as well as the lighter lines, so as to shorten your work. Explain carefully, as in a letter to a friend, why you made the choices you did, and how those choices help to make the counting process for the product more efficient.

FIGURE B6. Assessment 1, p. 1. Mathematics 305, Fall Semester 1997.

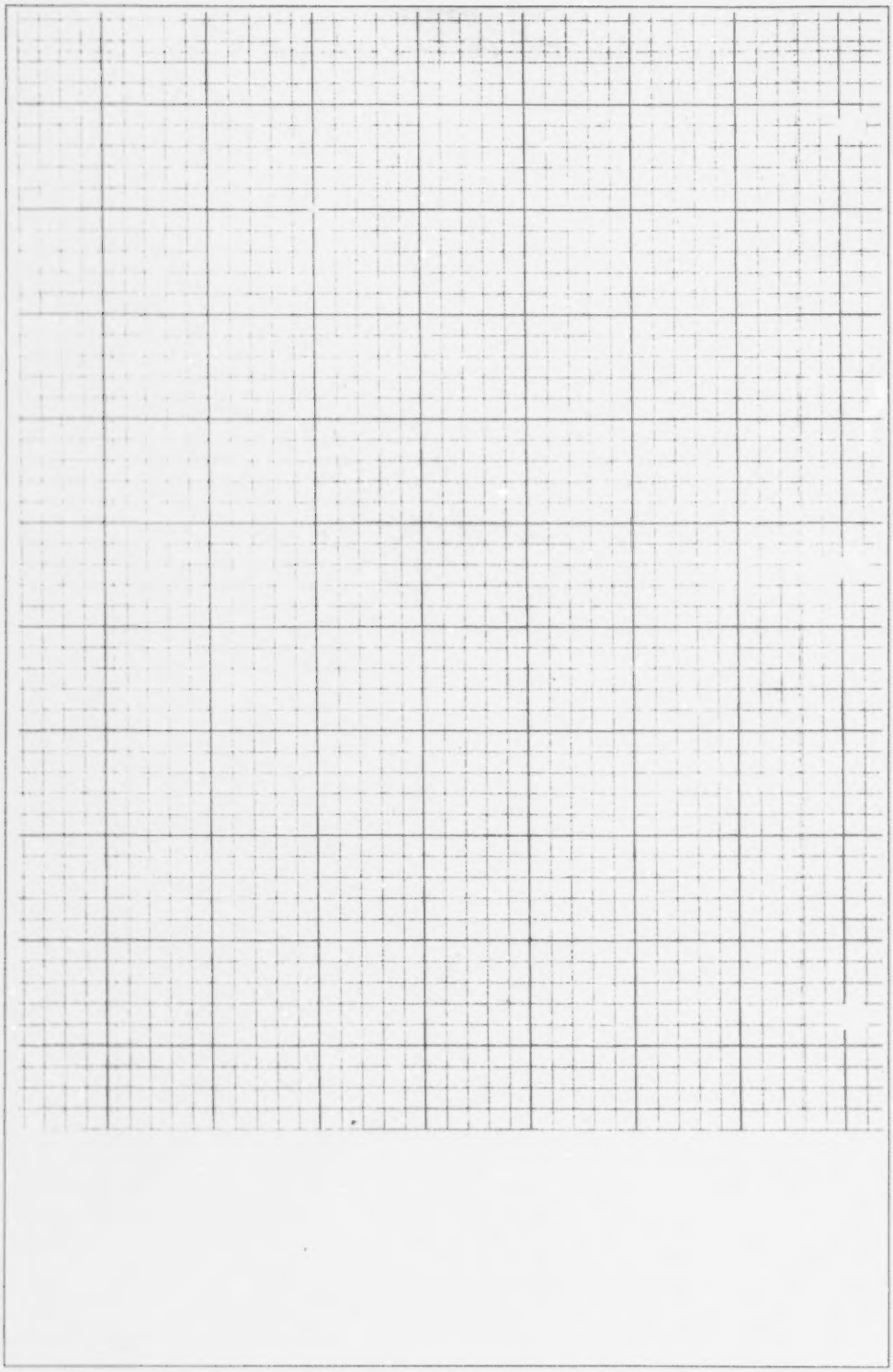


FIGURE B7. Assessment 1, p. 2. Mathematics 305, Fall Semester 1997.

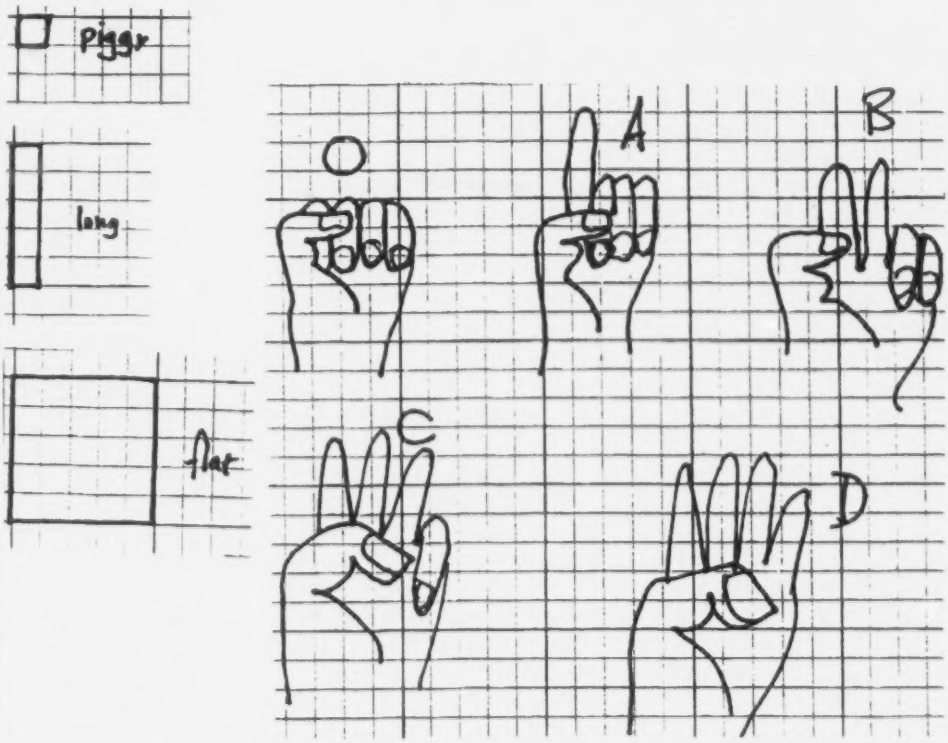
BEST COPY AVAILABLE

To: Math 305 students
From: Professors Speiser and Walter
Subject: Task 5 - Appeal to your imagination
Date: October 13, 1997

Our goal is to create a brave, new world, in which we can count, write numbers and do meaningful arithmetic, without leaving that world. Are you ready?

Task: In each group, use the blocks below, including "solids" if you wish, to come up with a number system that only uses the symbols O, A, B, C and D. Use the blocks to help develop and explain your system. Your number system should represent any number, no matter how large, and allow you to add, subtract, multiply and divide.

Rules: Never even think in base 10. Always express yourself with blocks, with the symbols O, A, B, C and D, or with the hand gestures associated with those symbols



[Adapted from Restructuring Mathematics Education by D. Schifter and C. Fosnot]

FIGURE B8. Assessment 3. Mathematics 305, Fall Semester 1995.

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Math 305 Professors Speiser and Walter October 8, 1997

Assessment 2 (30 minutes)

First, draw a picture to show how to multiply 25 by 34 using the standard base 10 blocks (piggies, longs, flats, solids). How does your work display what multiplication of whole numbers means? Second, use your work above to explain to what extent this diagram, in the spirit of an Italian arithmetic book from the fifteenth century, accomplishes the same purpose.

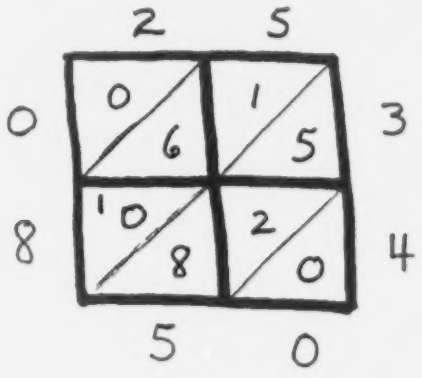

FIGURE B9. Assessment 2. Mathematics 305, Fall Semester 1997.

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To: Math 305 students
From: Professors Speiser and Walter
Subject: As few blocks as possible
Date: October 6, 1997

In the tasks below, a piggy represents the number 1.

1. First, pick up a solid. Then take away 17 piggies, in such a way that the difference will be represented by as few blocks as possible. Do this as simply as you can. Of course, you'll quickly find good ways to do this. Now, however, how might you predict your steps? In your explanation, please be sure to say exactly what subtraction means here, and exactly how that meaning is expressed, both by your actions with the blocks, and by your predictions.

2. Represent 36 and 42 with blocks, using as few blocks as possible. Then, starting with these representations, represent their product, as simply as you can. Now for the main question! Look at how you organized your product. How might you predict, without counting directly, how many flats, longs and piggies there will have to be? And finally, how might you use your predictions in order to find the product?

FIGURE B10. Task 6. Mathematics 305, Fall Semester 1997.

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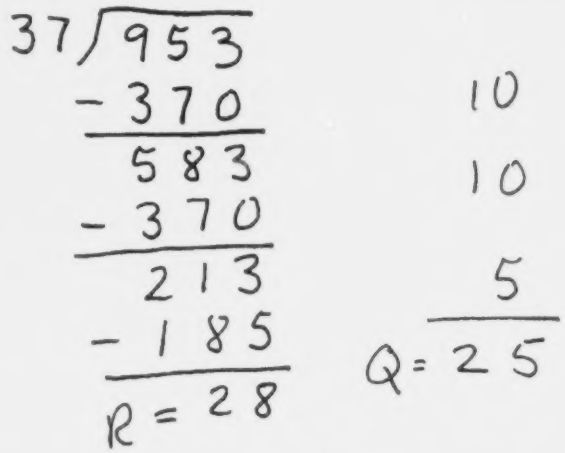
Math 305 Professors Speiser and Walter October 4, 1995
(Oct. 27, 1997)

Assessment 3 (30 minutes)

Here is a child's work on a division problem. Is she right? A colleague thinks that this child fails to understand the concept of division. Based on the evidence at hand, what would you tell him?



Please give your response on the other side of this page. Use pictures if appropriate to help support your words.

FIGURE B11. Assessment 3. Mathematics 305, Fall Semester 1997.

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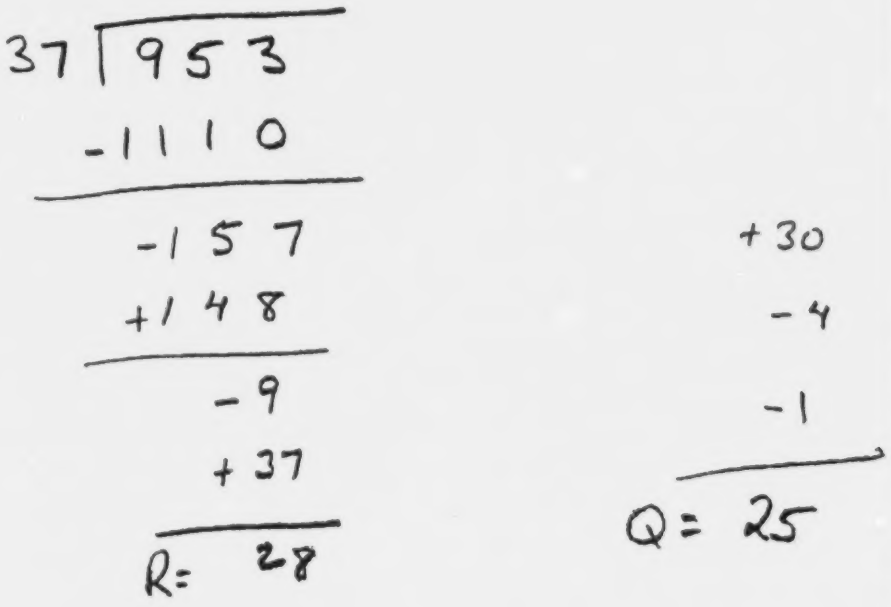
To: **Math 305 Students**
From: **Professors Speiser and Walter**
Subject: **Task 7: Introducing Zoe**
Date: **October 27, 1997**

Zoe's a friend of ours. Right now, she's ten years old. Here's a sample of her work.



Is she right? Her teacher feels that Zoe's work here looks crazy. What might you say to Zoe's teacher?

Just to catch Zoe's spirit, can you work out a new procedure <u>of your own</u> for whole number division? How might you explain it?

FIGURE B12. Task 7. Mathematics 305, Fall Semester 1997.

To: Math 305 Students
From: Professors Speiser and Walter
Subject: Task 8: Postman Stories*
Date: October 29, 1997

We would like to invent an arithmetic for numbers with signs. To do this, we will imagine working in a particular setting and how we might represent what occurs.

1. Suppose that along with your other mail, such as letters from home, your postman from time to time brings you checks and/or bills. Let's designate these special pieces of mail in a particular way. For example, let's symbolize a **check** for \$10 by $^{+}10$ and let's write $^{-}8$ to represent a **bill** for \$8.

   a) On Tuesday, the postman brings a check for \$11 and a bill for \$6. Are you richer or poorer after the postman's visit on Tuesday? By how much? Why? How might you represent, symbolically, Tuesday's visit by the postman?

   b) Can you make a postman story for: $^{+}13 + {}^{-}8$?

   c) What other kinds of postman stories and associated representations can you construct?

   d) If the postman brings a check for \$25, a check for \$7, a bill for \$10 and a bill for \$3, are you richer or poorer? By how much?

2. Suppose the postman can also take away checks and/or bills.

   a) The postman comes on Monday. He says,"I'm sorry, the check I left last week was really for the man next door." He takes back the check (which happens to be for \$18). Does his visit on Monday make you richer or poorer? Explain your response?

   b) On Wednesday the postman brings a check for \$11. He also says that the bill for \$5, which he brought a week before, was really for the family upstairs, so he takes it away. Does his visit on Wednesday make you richer or poorer? By how much? Explain. How might you represent, symbolically, the postman's Wednesday visit?

   c) Write a postman story for: $^{+}25 + {}^{-}15 - {}^{-}10$.

3. Design postman stories which, you feel, would be essential in building an arithmetic for numbers with signs.

*Adapted from Robert B. Davis (1980), *Discovery in Mathematics: A Text for Teachers.*

FIGURE B13. Task 8. Mathematics 305, Fall Semester 1997.

To: Math 305 Students
From: Professors Speiser and Walter
Subject: Pizzas[1]
Date: November 7, 1997

Four children–one of them is Zoe of course—meet at the Boun Gusto Pizzeria every Thursday evening. Each time they go, they share 3 pizzas. The kitchen is often a little busy, so the pizzas might arrive in different combinations.

1. The kitchen is extremely busy, so the pizzas come out one at a time, with some time in between. How might the children share them?

2. Less busy. Two pizzas come out together, and the third one somewhat later. Same question.

3. Nearly empty house. This time the pizzas arrive together. Same question.

4. Another time, eight children—one of them Zoe, of course—shared 5 pizzas, in such a way that each child got 1/4 + 1/4 + 1/8. Can you guess how the pizzas were served?

[1]Adapted from L. Streefland, *Fractions in Realistic Mathematics Education* (Dordrecht, 1991).

FIGURE B14. Task 9. Mathematics 305, Fall Semester 1997.

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To: Math 305 Students
From: Professors Speiser and Walter
Subject: Beads
Date: November 7, 1997

If we divide, say, 3 pizzas among 4 children, we know the "how many" (namely three) and the question is to find the "what", the amount of pizza for each child. Here's another example of this kind.

1. I have beads, and 18 make up one foot. How long is each bead?

Now, we go the other way, knowing the "what", and asking for "how many."

2. I want to make a necklace for Zoe, using these beads. She'd like her necklace to be 1 1/2 feet long. How many beads?

Notice that you might solve this problem in a number of ways. Can you find some common principles?

FIGURE B15. Task 10. Mathematics 305, Fall Semester 1997.

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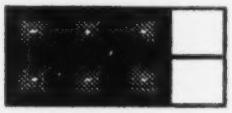
To: Math 305 Students
From: Professors Speiser and Walter
Subject: Chocolate[1]
Date: November 7, 1997

You are given 3/4 of a chocolate bar.



A whole bar costs $1.20. How much is the shaded portion worth?

[1] Adapted from L. Streefland, *Fractions in Realistic Mathematics Education* (Dordrecht, 1991).

FIGURE B16. Task 11. Mathematics 305, Fall Semester 1997.

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To: Math 305 Students
From: Professors Speiser and Walter
Subject: Task 9: How big are they?
Date: November 5, 1997

1. Cuisenaire rods can represent fractions. Try your hand!

If the light green rod is 1/3, what's the blue rod?
If the blue rod is 1/3, what's the light green rod?
If the blue rod is 1/2, what's the red rod?
Can you find two rods, one of which represents 1/2 while the other represents 1/3?
Can you find two rods, one of which represents 3/5 while the other represents 4/5?
Construct two more questions in this spirit, and try to answer them.

2. Which fraction is bigger, and why?

1/2 or 1/3
3/5 or 4/5
5/6 or 6/7
8/9 or 9/10
999/1000 or 1000/1001

3. Look back at your reasons. Might they give the beginning of an informal theory about when one fraction is bigger than another? Try to organize a small set of guiding ideas which you think may have a chance to work for any two fractions, and make sure they work, at least for the examples above. Then try your theory on the following examples, refining your ideas as needed:

Which is bigger, and why?

10/21 or 11/20
11/20 or 10/19
16/23 or 15/22
18/25 or 17/24
12/11 or 13/12

4. Might Cuisenaire rods, used suitably, help to answer questions in this last group? Might they help to justify your theory? Are there other ways of representing fractions which could help here? Please explain.

FIGURE B17. Task 12. Mathematics 305, Fall Semester 1997.

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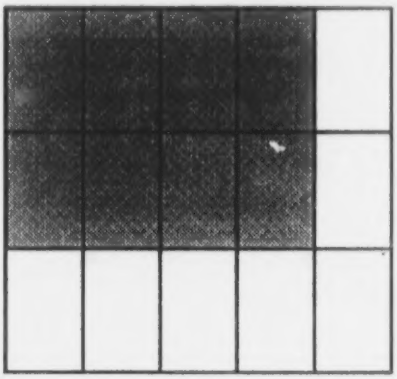
Math 305 Professors Speiser and Walter November 12, 1997

Assessment 4 (30 minutes)

1. Zoe thinks big. Here is how she multiplies 2/3 by 4/5.



Zoe explains that her concept of multiplication is finding a rectangular area. For her, the large square represents a square mile. In particular, the sides of the large square are one mile long, so that the shaded rectangle has dimensions 2/3 and 4/5 miles, respectively. The area of the shaded rectangle, she explains, is therefore $2 \times 4 = 8$ fifteenths of a square mile. She further points out that this area is both 2/3 of 4/5 and 4/5 of 2/3. Are her claims correct? Can you explain, using her picture?

2. Now for some big thinking of your own. Can you find the sum of 2/3 and 4/5, and justify your conclusion, by cutting giant squares in pieces? Please explain.

FIGURE B18. Assessment 4. Mathematics 305, Fall Semester 1997.

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Math 305 Professors Speiser and Walter November 19, 1997

Assessment 5 (30 minutes)

You walk for 1 hour and 40 minutes, at a speed of, say, 2 and 2/5 miles per hour. How far do you go?

We have a rectangle whose length is 1 and 2/3 miles and whose width is 2 and 2/5 miles. What is its area?

Zoe says these questions are so related that they have to have the same answer. A colleague thinks that the two questions are fundamentally different, and therefore that Zoe has been beguiled by the coincidence that both, in his view, require multiplication. What might you explain to him?

FIGURE B19. Assessment 5. Mathematics 305, Fall Semester 1997.

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To: Math 305 Students
From: Professors Speiser and Walter
Subject: Task 11: Decimals
Date: December 3, 1997

This task is about building decimal representations of fractions, We'll use the base ten blocks to help us here. Specifically, let's say that the solid represents 1. What does a piggy represent? A long? A flat?

1. How might you write the number represented by 7 piggies, 3 longs, 4 flats and 2 solids? How might you use the blocks to represent 1.428? 0.035? 0.06?

How might the blocks help you explore decimal arithmetic? The following questions may get you started.

2. Add 2.214 and 0.367.
3. What is 3.214 – 1.967?
4. How would you multiply 0.8 by 0.03? 3×1.36? $.72 \times 2.4$?
5. How might you find 1 5? 3 2? .1 5? .01 5? 1 4?

Questions for further investigation:

6. How would you write 1/2 as a decimal number? How about 1/4? 1/8?
7. Can you represent 1/3 as a decimal number? What's different here? How do you make sense of this new representation?
8. Try 2/3, 1/6, 1/11.

Zoe claims that 1= 0.9999... (here the 9 is repeated forever). How could this make sense? What do you think?

FIGURE B20. Task 13. Mathematics 305, Fall Semester 1997.

Math 305 Professors Speiser and Walter December 8, 1997

Assessment 6 (30 minutes)

Recently, while sitting in an airplane on the taxiway at the Cincinnati airport, waiting our turn to takeoff behind eight or so other aircraft, I heard the captain announce to the passengers that the delay was because the air traffic control was requiring a thirty five mile gap between successive aircraft at takeoff. Looking out my widow, I could see the aircraft in front of us roll onto the runway, pause and then start their takeoff, so I decided to time the interval between the next two aircrafts' takeoffs. The interval was exactly 2 minutes! Our flight from Cincinnati to Newark, NJ was exactly 89 minutes and the distance between the two cities is 569 air miles. Professor Speiser and I did some rough estimates in our heads that suggested that this information wasn't consistent. What do you think?

FIGURE B21. Assessment 6. Mathematics 305, Fall Semester 1997.

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Notes

1. The standard format of the course failed dismally and yet persisted for more than a decade, as a dissatisfied department, shifting toward pure research, withdrew resources. The authors, on the other hand, had volunteered—as concerned mathematicians—to collaborate with a nearby elementary school to rebuild its mathematics program, and now, after two years, the time seemed right to trigger, through our work in Mathematics 305-306, a discussion about teacher education that might bring about real change. As we write this, beginning our experiment's third year, the battle is still on.

2. This camera, an overworked and temperamental 8mm Sony, was often run by our graduate assistant Tara Lewis, who soon became a full and highly valued partner in our research enterprise. In particular, her choice of what to film and how to film it made this study possible.

3. We used engineering paper, which is squared with fine and heavy lines, each fifth line heavy.

4. Here note Maher and Speiser (1997a, 1997b) as well as Speiser (1997).

5. It plants the seeds for combinations, for example.

6. There was one brief gap, perhaps a minute long, due to a camera malfunction.

7. Here, for the first time in our research, we consciously followed and supported an emergent discourse in a mathematics classroom (Speiser & Walter, 1997).

8. See Maher and Martino (1996a, 1996b), Maher and Speiser (1997a, 1997b, 1998), and the related works these sources indicate.

9. With deepest thanks to the members of the Rutgers Long-Term Study, especially Carolyn Maher, Bob Davis, Alice Alston, Amy Martino, Elena Steencken, and Lynda Smith.

10. What Latour (1987) calls "closing a black" box seems similar to what Davis (1984) refers to as "achieving noun status." While Davis's term focuses on the changing status of a piece of knowledge, Latour prefers to emphasize the actors and the setting. The converse process, opening a black box, is, similarly, a pointer to the actors and their setting, as they reconsider knowledge that no longer may be taken as securely known.

11. The thesis of our graduate student, Tara Lewis (1997), studies a later segment, based on a preliminary versions of the framework we use here. We are very grateful for Tara's efforts, which helped clarify a number of important issues.

12. For example, Allen Newell and Herbert Simon (1972).

13. This term has been so often used and in so many senses that we need to be precise in saying which sense, exactly, we intend. In particular, Davis (1984, p. 71) suggests, "Avoiding confusion may be somewhat easier if one remembers that Minsky's *frame* is always a special kind of knowledge representation structure, stored in memory and retrievable from memory."

14. In this paper, we use words like "set" and "subset" quite informally, as parts of everyday language, simply to indicate collections of objects. In this section, for simplicity, we restrict attention to finite sets.

15. We postpone default evaluations until we have described the subframes.

16. As each element is dealt, appropriate subframes of the dealing subframe can keep track of where the elements of S have gone so far and what still remains to be dealt. These subframes are related by what Minsky describes (1975, pp. 216–219) as "frame transformations." Each subframe shows a different distribution of the elements of S to a fixed set of terminals, and the transformations describe only what changes as we move from one subframe to the next.

17. Clearly this step connects to the binary partition frame.

18. Here we are definitely talking about "information about how to use the frame," which seems connected, here, to the default assumptions.

19. In Minsky's terms (1985), at two contiguous *level bands*.

20. The block that we call a "piggy" is often called a "unit," which seems to constrain this block to represent the number 1. Hence we resist this terminology. If instead we make the solid represent the number 1, then the smaller blocks can be used to represent decimals. The piggy, for example, would then represent .001. This freedom to interpret leads to depths of extra meaning. Without a standard term to lean on, we invited our students to suggest a useful name for the smallest block. After some discussion, they proposed "piggy," as in the children's counting rhyme "This little piggy went to market, this little piggy stayed home,..." Our students' choice seems truly excellent, because in practice we tend to use the piggy as a basic counter, in comparison to larger blocks.

21. We will tacitly assume that collections of real blocks are finite.

22. The *default* evaluation.

23. If we want to count in 45s, however, we could assign the value 45. This leads to very interesting algorithms, as our students shown us.

24. In base 10, the quotient, given by 41 piggies, is represented by 41 /125 = .328.

25. Or LEGO blocks, for that matter (Papert, 1993a, 1993b; Martin, 1992), which probe, among other things, some of the number theory underlying the blocks' dimensions.

26. As Ginsburg (1989) emphasizes, fundamentally.

27. But perhaps not uniquely.

28. The default evaluation, at least for us.

29. This illustration and the surrounding discussion date back to a series of informal workshops and conversations (Speiser, 1971) at the East Hill School in Ithaca, New York, where the first author taught for one year. Both authors join to acknowledge thoughtful contributions by several teachers there, including Marie Matthews, Bill Mutch, Cindy Walters, and Ruth (now Heather) Wishik.

30. This independence, as we see it, seems to draw upon concrete, detailed experience in early childhood, not formal proof. As in the counting construction frame, we consider each choice of matching to be just as good as any other, accepting all with prejudice toward none.

31. The number system is infinite, but T is not, so we can be sure that the process stops with all members of T participating in the match.

32. Or, at times, the toes of one small foot.

33. A pun on "body odor"?

34. A further pun?

35. Children's slang for "Look here!" or "Look at that!"

36. We lost several minutes here because our camera jammed.

37. Is this a gentle pun on Nancy Reagan's "Just say NO" to drugs?

38. Here the camera followed another group for perhaps eight minutes.

39. In particular, we find no motivation to postulate either "a problem space" or a consensus. This group, in particular, succeeds, we feel, because its members seem so strikingly distinct and individual.

40. See Note 33.

41. See Note 34.

42. See Note 35.

43. See Turner (1967, 1974, 1988, 1990) and Meyerhoff (1979), who broaden the scope of investigation—originally redressive rituals in tribal cultures—to include many kinds of social dramas, such as stage plays, political confrontations, and religious pilgrimages. Turner recognized that close study of conflicts, as in drama, can reveal both structure (as otherwise unspoken background) and change (seen in relation to that background).

44. The text needs to be distinguished from the transcript of our students' dialogue.

45. The five we study here. The range of number systems that our initial task allows, like Schifter and Fosnot's (1993), is wider and goes far beyond our current frame descriptions.

46. She wrote this the next day.

47. See Appendix A.

48. Note that $D \times B = AC$, hence Holly's objection.

49. Pirie and Kieren (1992).

50. Here we follow Epston and White (1990), who propose a psychotherapy of narrative, based in part on text analogies and in part on Turner's view of social drama. A key step for these authors, in their role as therapists, is to help their clients to "externalize the problem" that the clients have presented, by examining *its* impacts and *its* needs.

51. This term, introduced by van Gennep (1908/1960) to describe the central phase of rites of passage (in which initiates are segregated, stripped of status and connections, to experience deep transformational events in settings that demand wide ranges of expression and performance) has taken on, in Turner's hands, quite general significance.

52. In the sense of Pirie and Kieren (1992).

53. Here, we paraphrase Latour (1987), but the earlier Lakatos (1976), which emphasizes refutations through examples as a key ingredient in building theories based on proofs, seems just as pertinent.

54. How we got permission and resources, a second social drama, would probably require another book to tell.

55. Now we use one-inch square builders' tiles, which work at least as well.

56. We were inspired by De Pree (1993), who writes with awe and wonder of his granddaughter, also named Zoe. He tells later (De Pree, 1998) how her mother named her.

57. Toward the end of the first semester, one student, who sat near the back of the room, stood up to say, "You want to know who Zoe is? OK, I'll tell you. *I* am Zoe." She has corresponded with us ever since.

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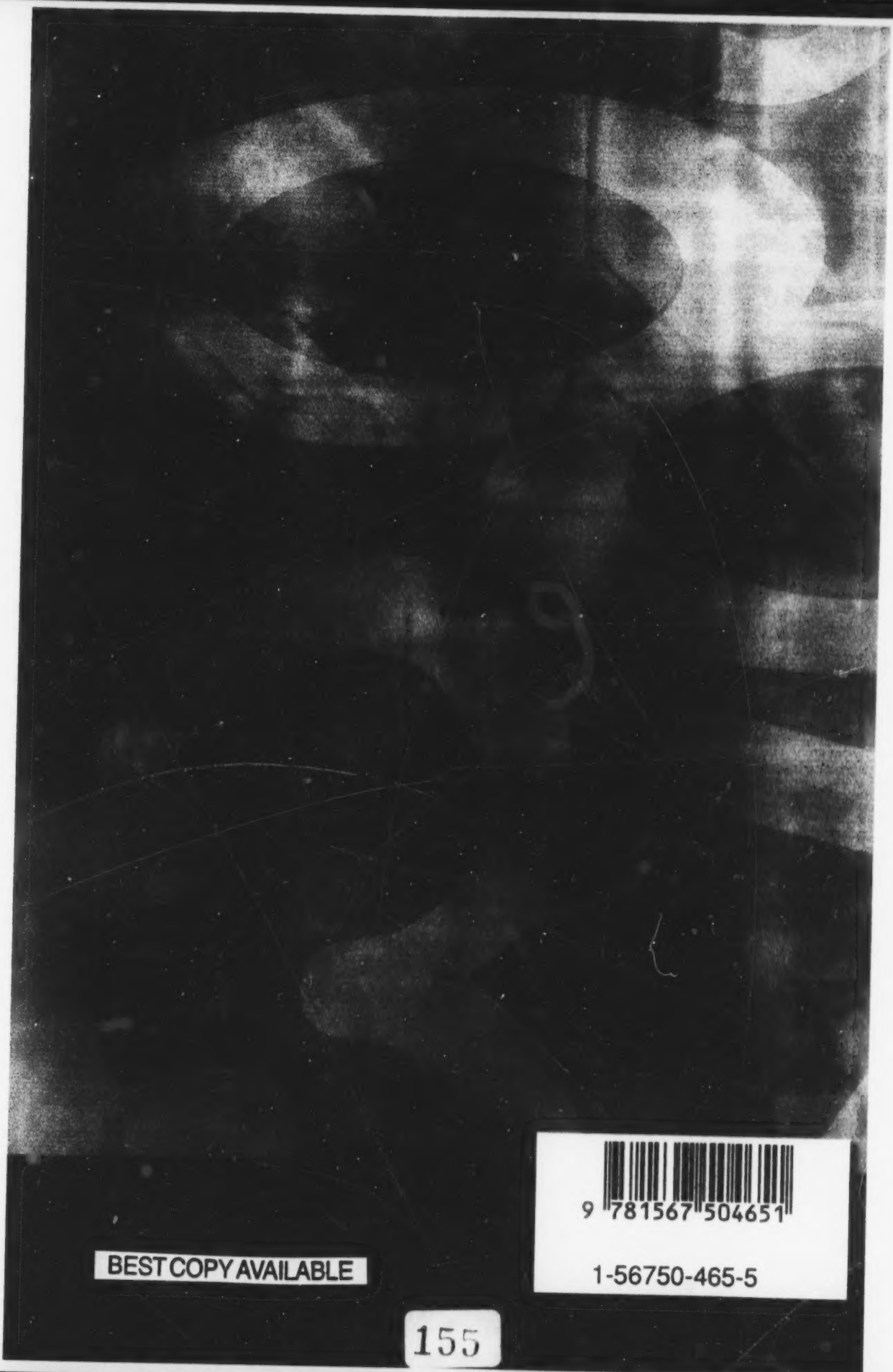

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